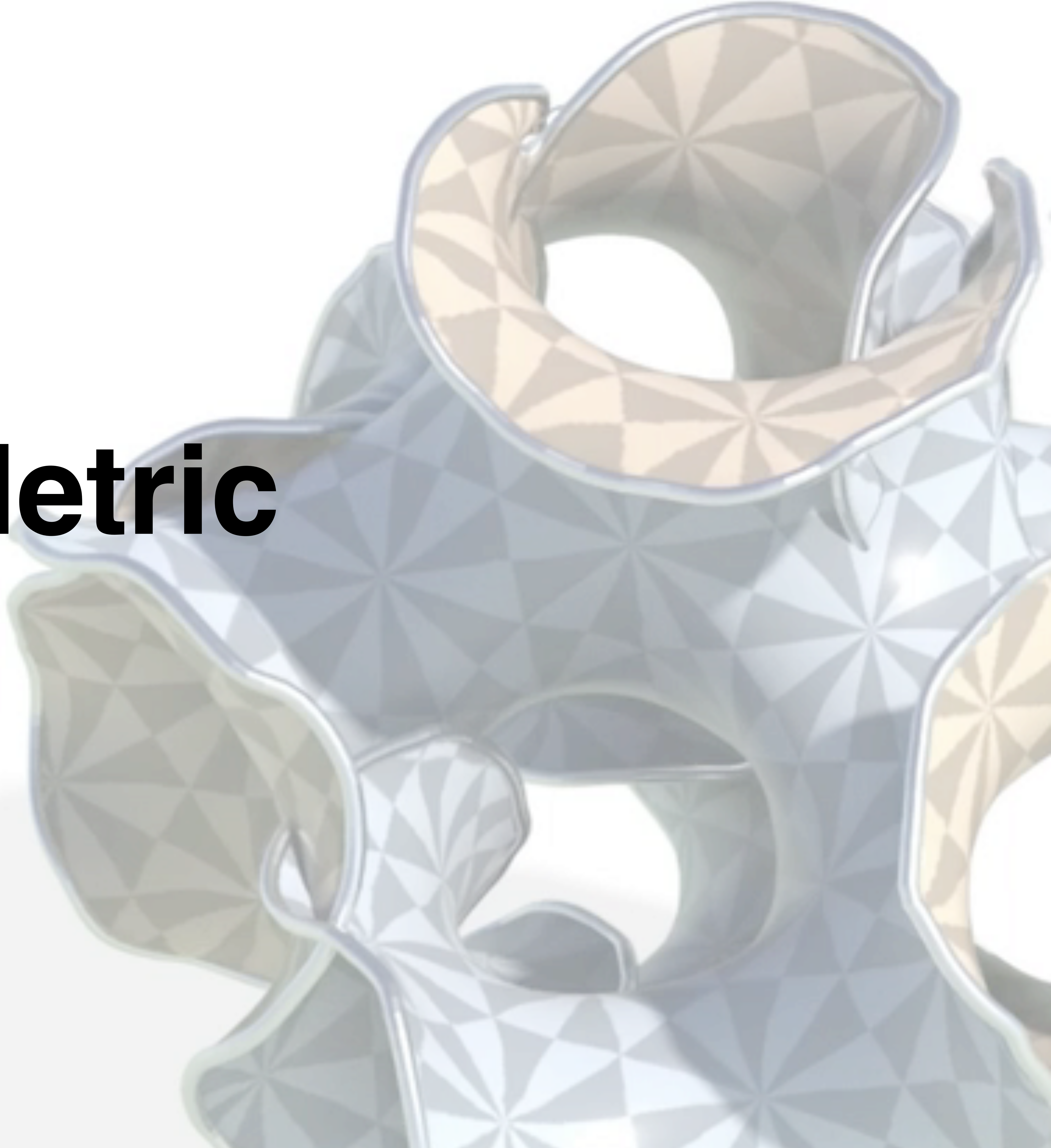
ICERM Illustrating Geometry and Topology
Sep 19, 2019

Shape from Metric

Albert Chern
TU Berlin / UC SanDiego



Shape from Metric

Albert Chern
TU Berlin / UC SanDiego

Felix Knöppel
TU Berlin

Franz Pedit
UMass Amherst

Yousuf Soliman
Caltech

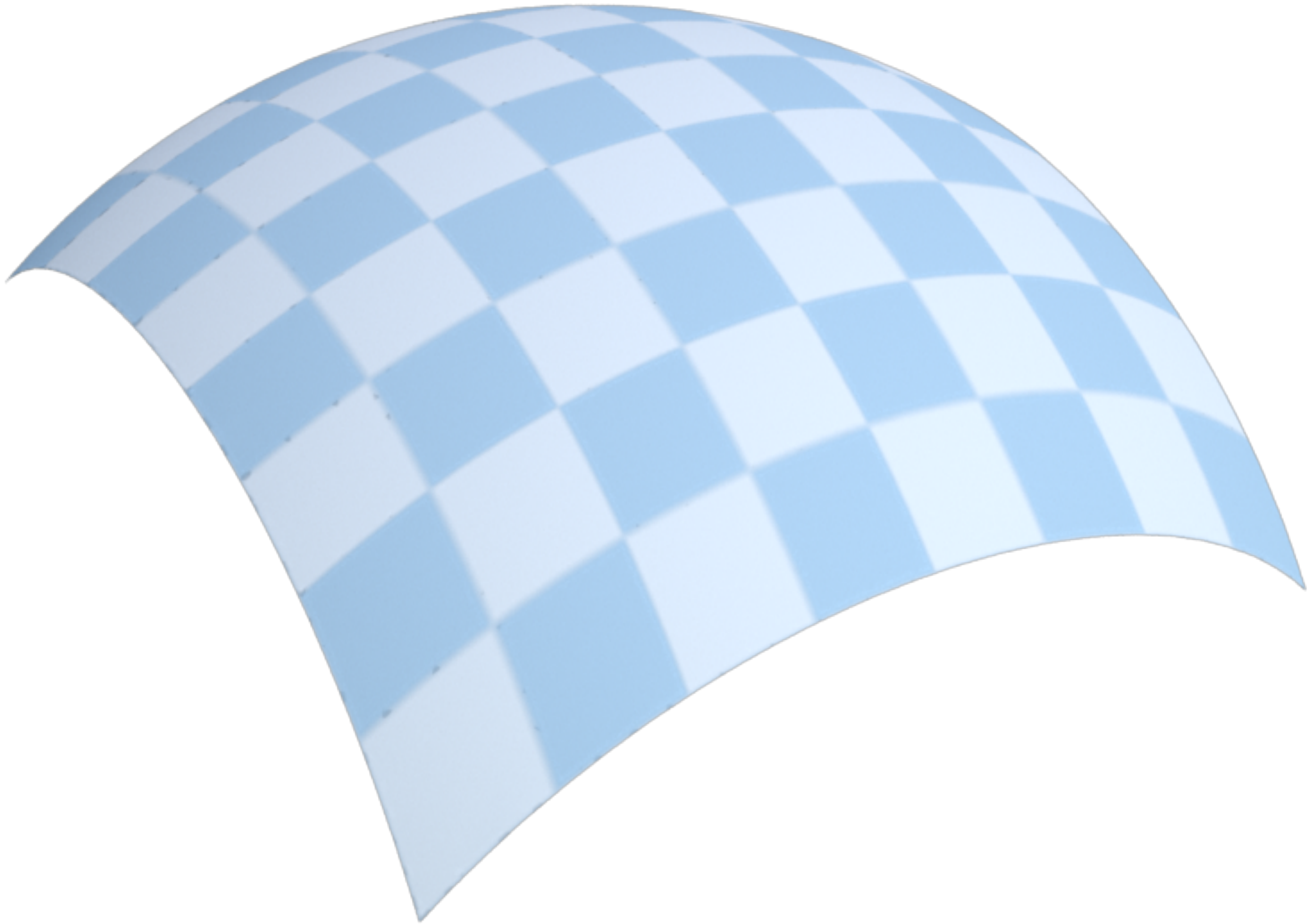
Ulrich Pinkall
TU Berlin

Peter Schröder
Caltech

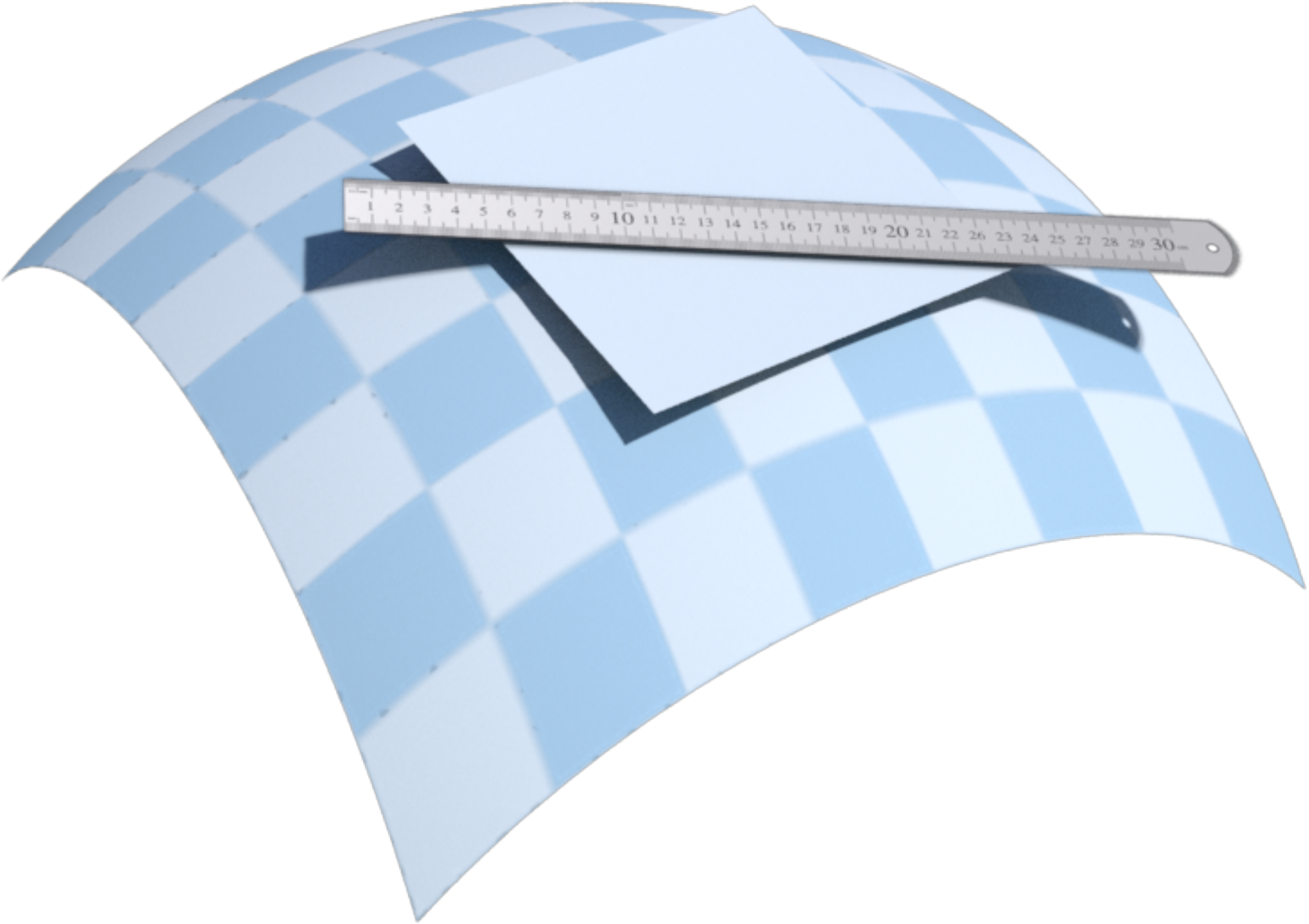
Olga Diamanti
TU Berlin

SFB TRR 109 Discretization in Geometry and Dynamics
EINSTEIN Foundation.de
Deutsche Forschungsgemeinschaft

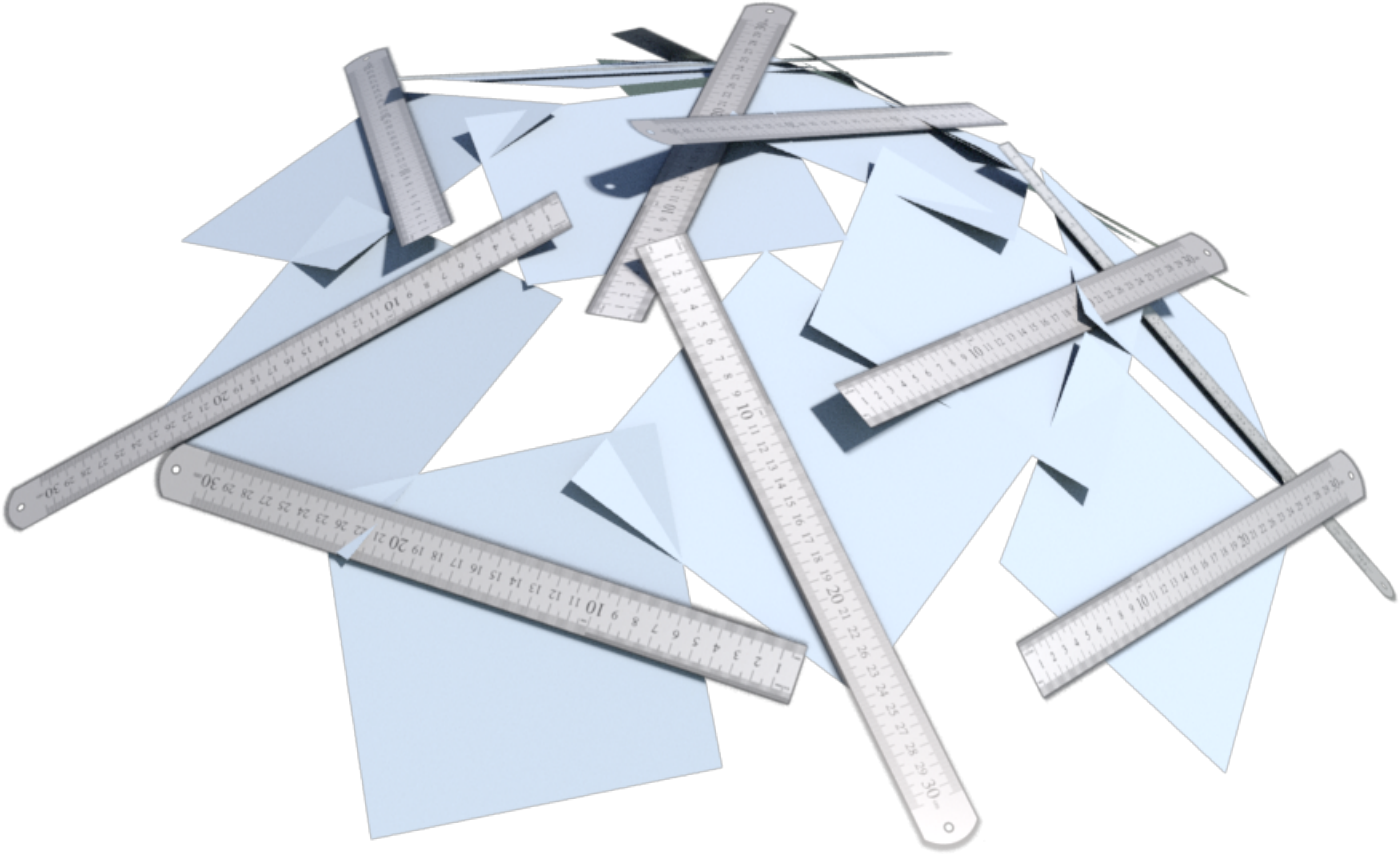
Differential geometry



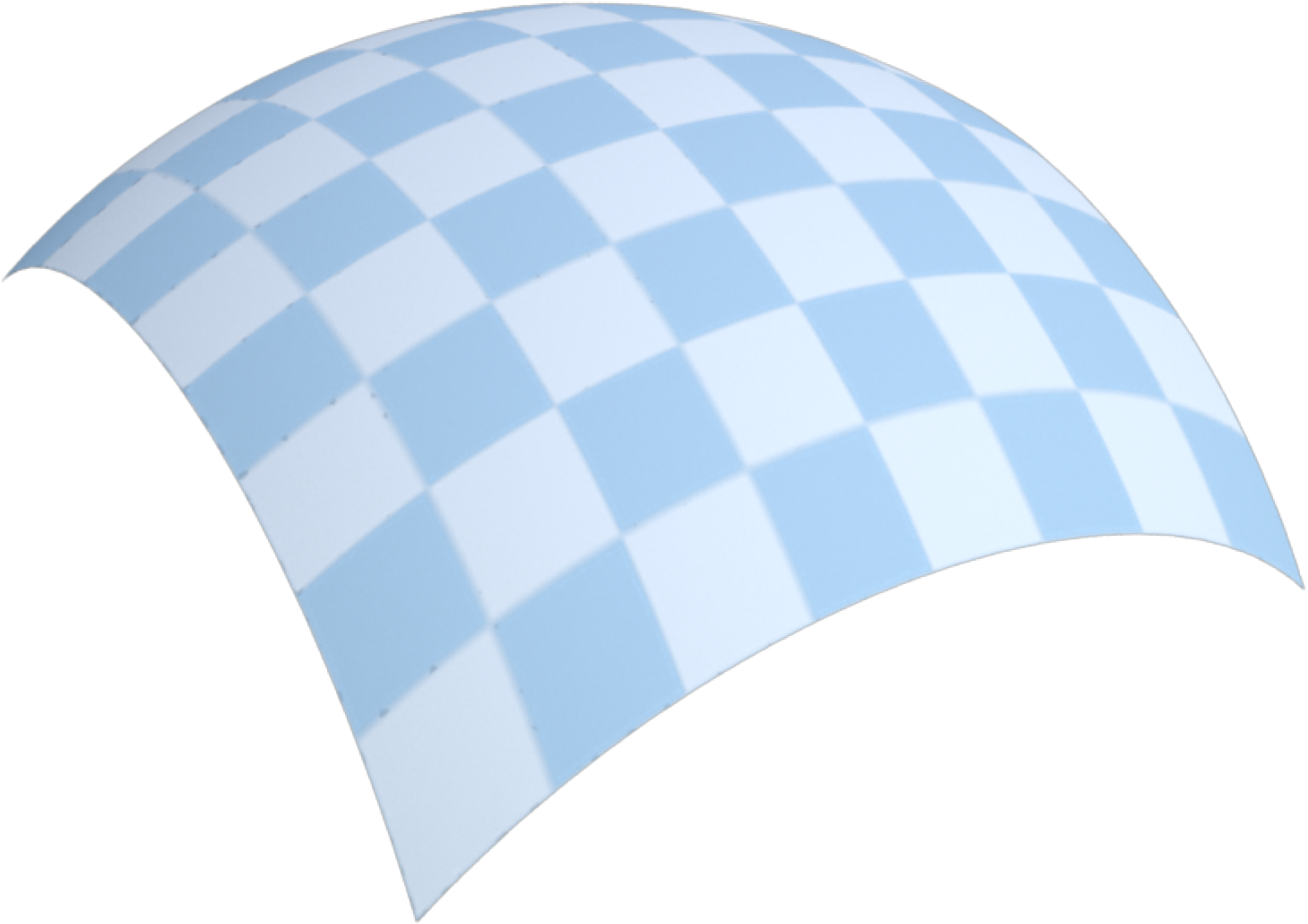
Differential geometry



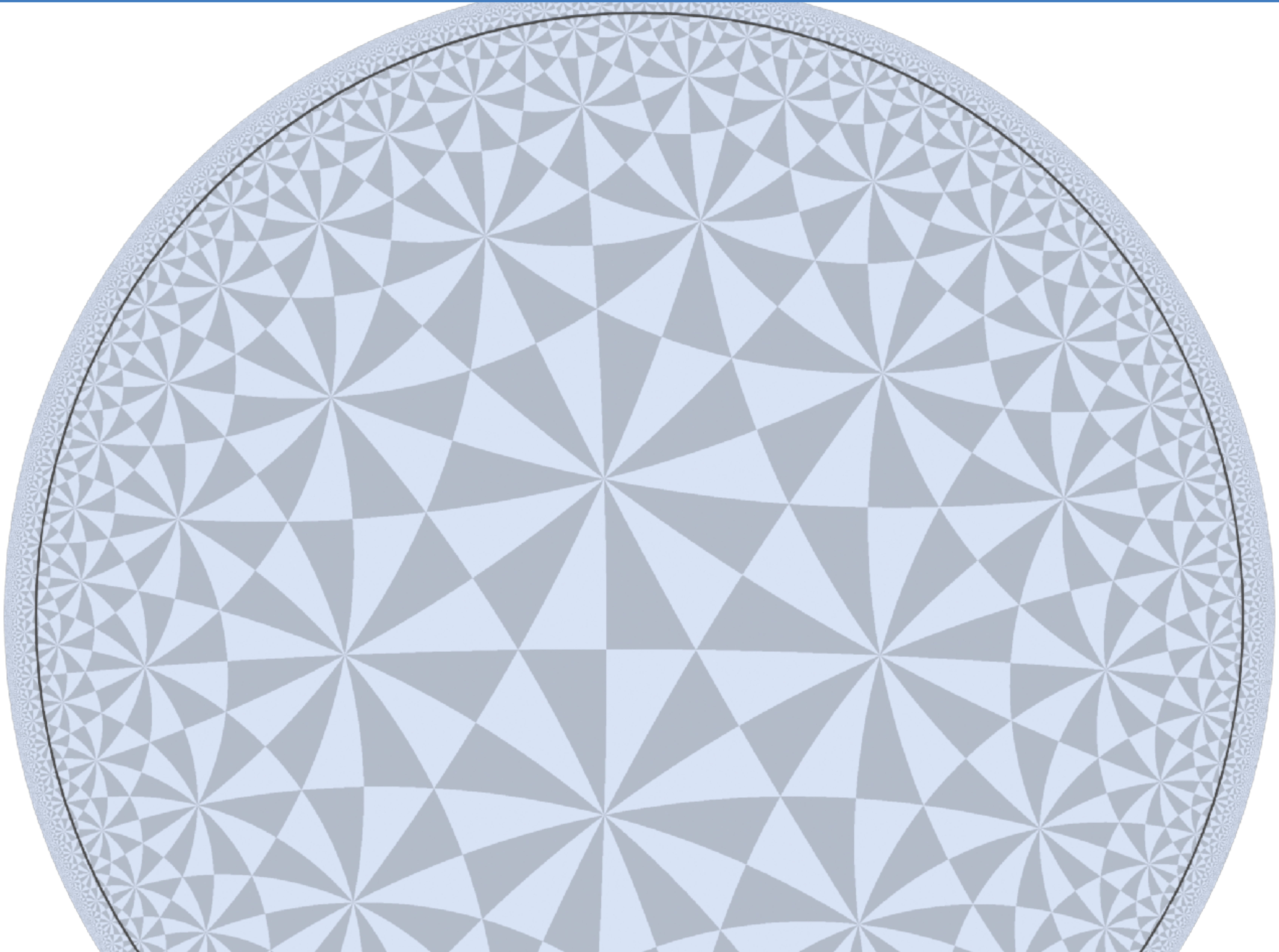
Illustrating differential geometry



Illustrating differential geometry

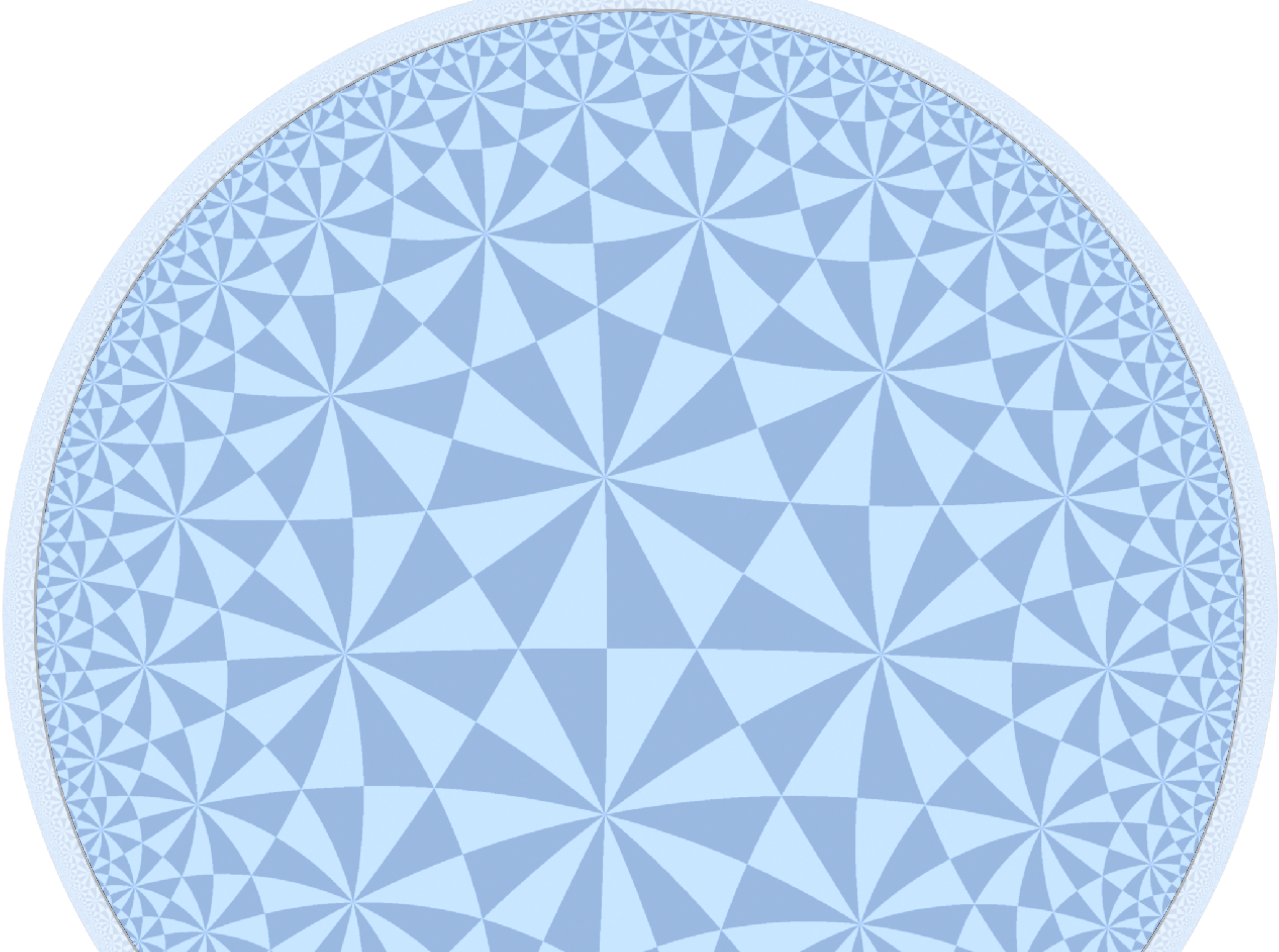


Mathematical visualization

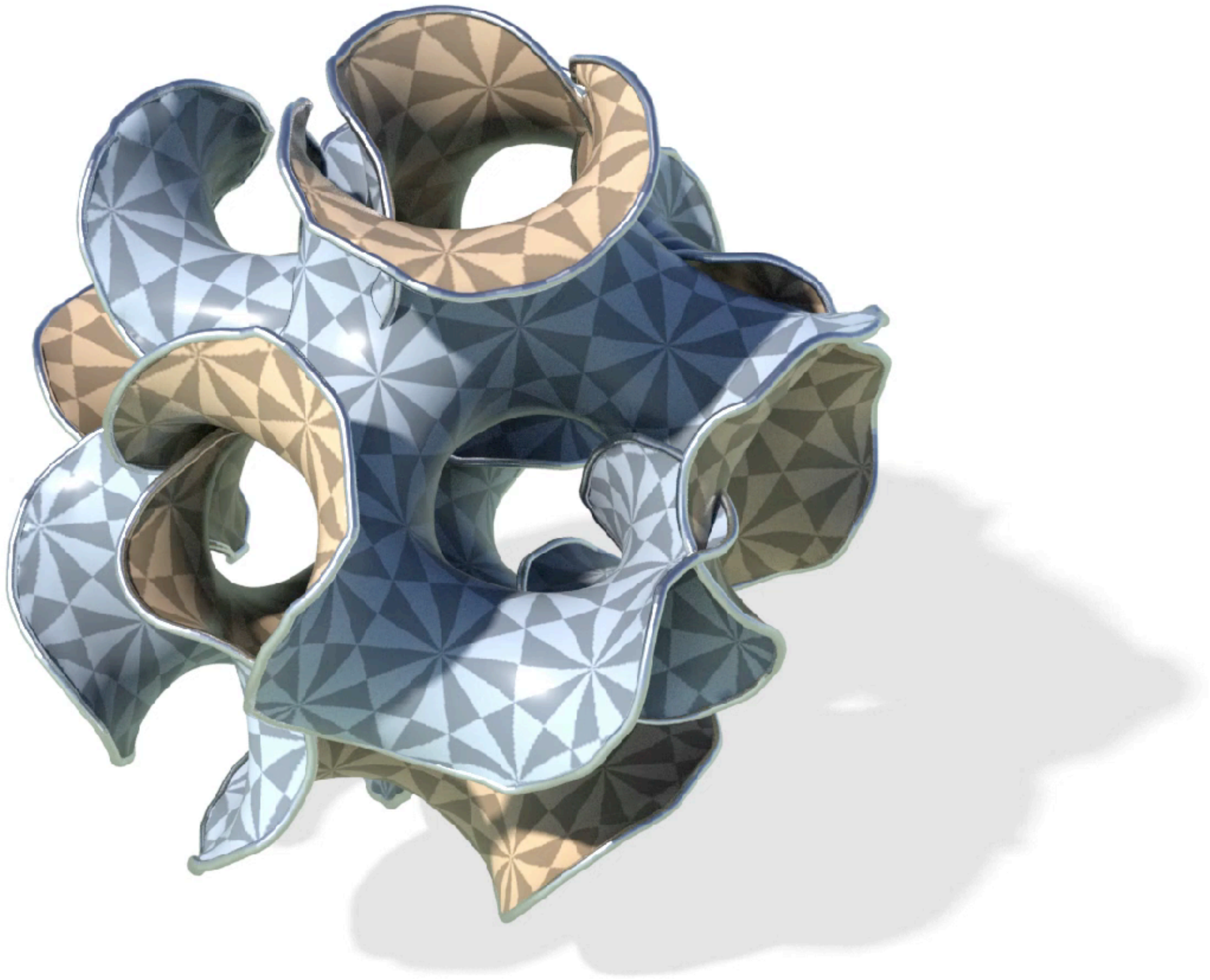


Mathematical visualization

Hyperbolic disk



Mathematical visualization



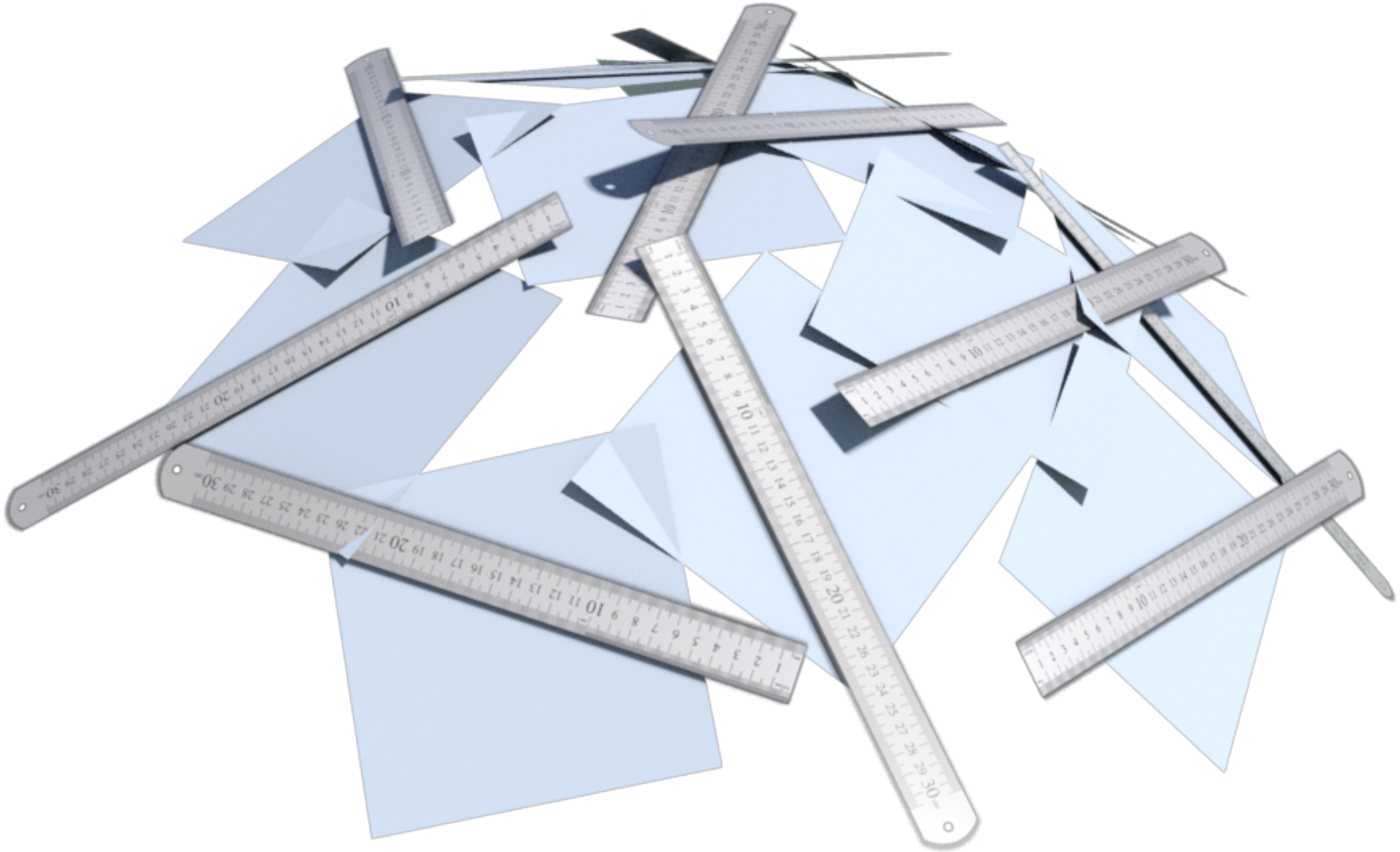
Local properties dictates global shapes

Local properties dictates global shapes

Shape from Metric

Differential property

e.g. Riemannian metric



Shape from Metric

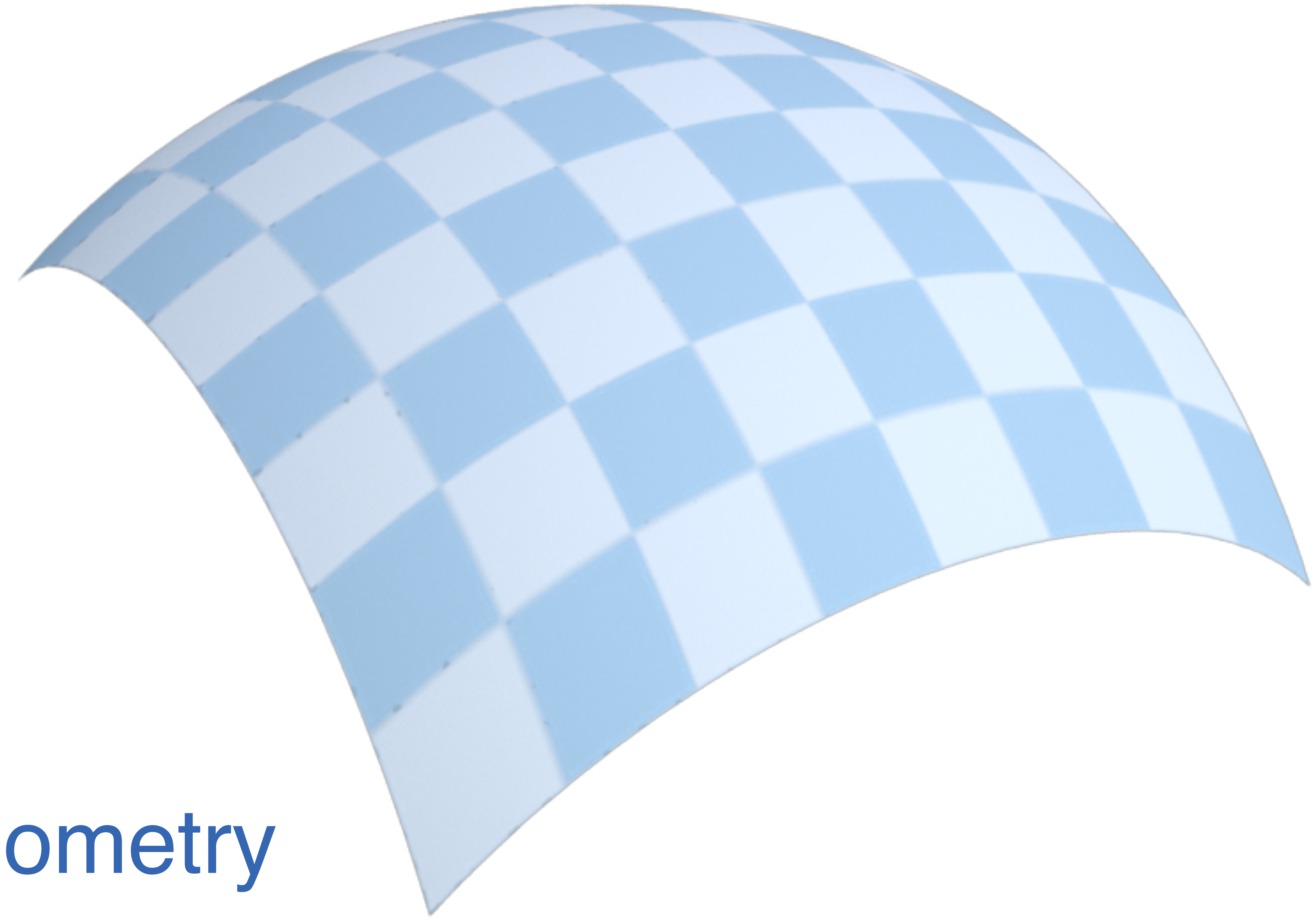
Differential property

e.g. Riemannian metric

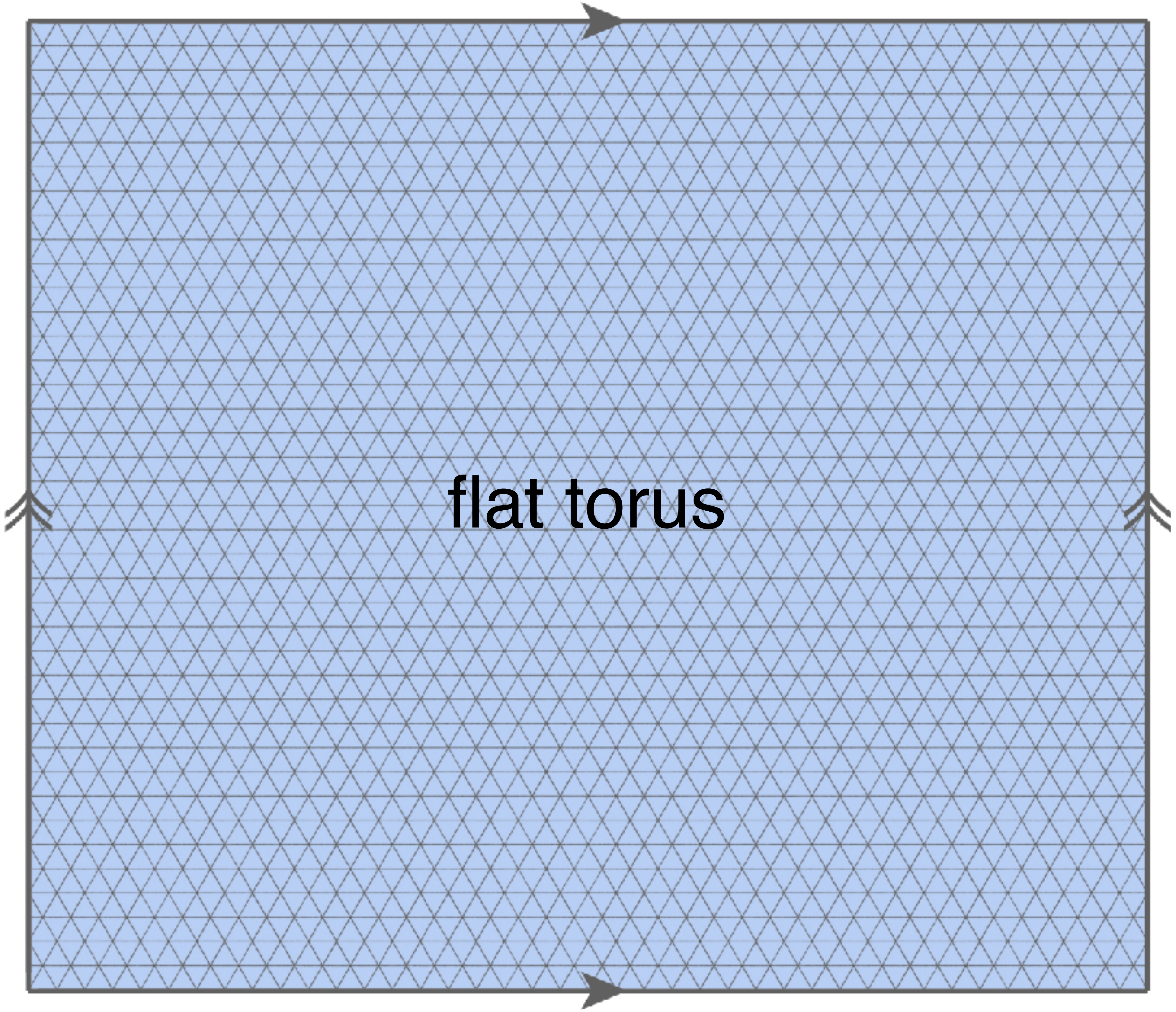
↓

Surface

best displays the intrinsic geometry
at the macroscopic level

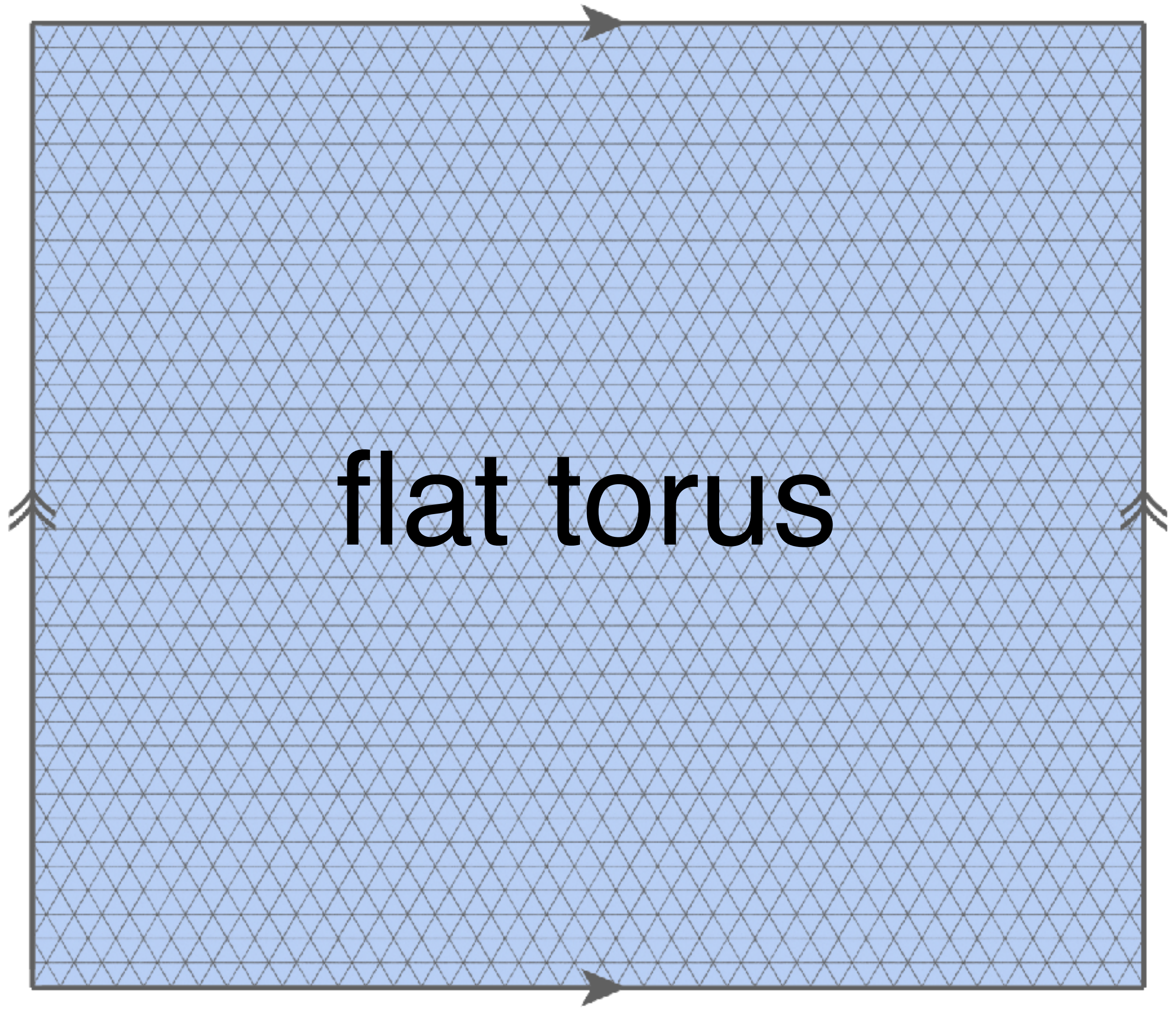


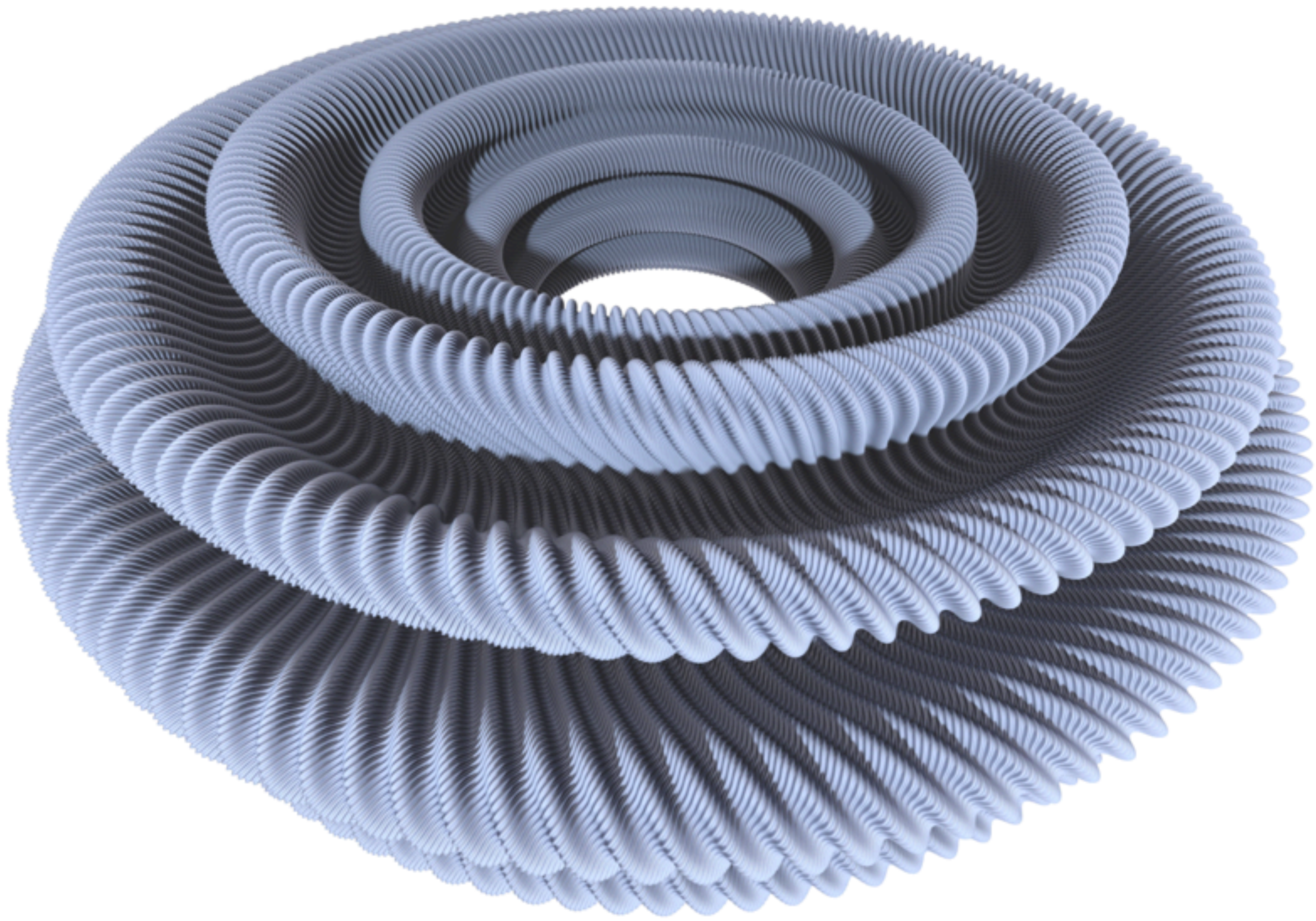
Flat torus



Flat torus

C^1 embedding

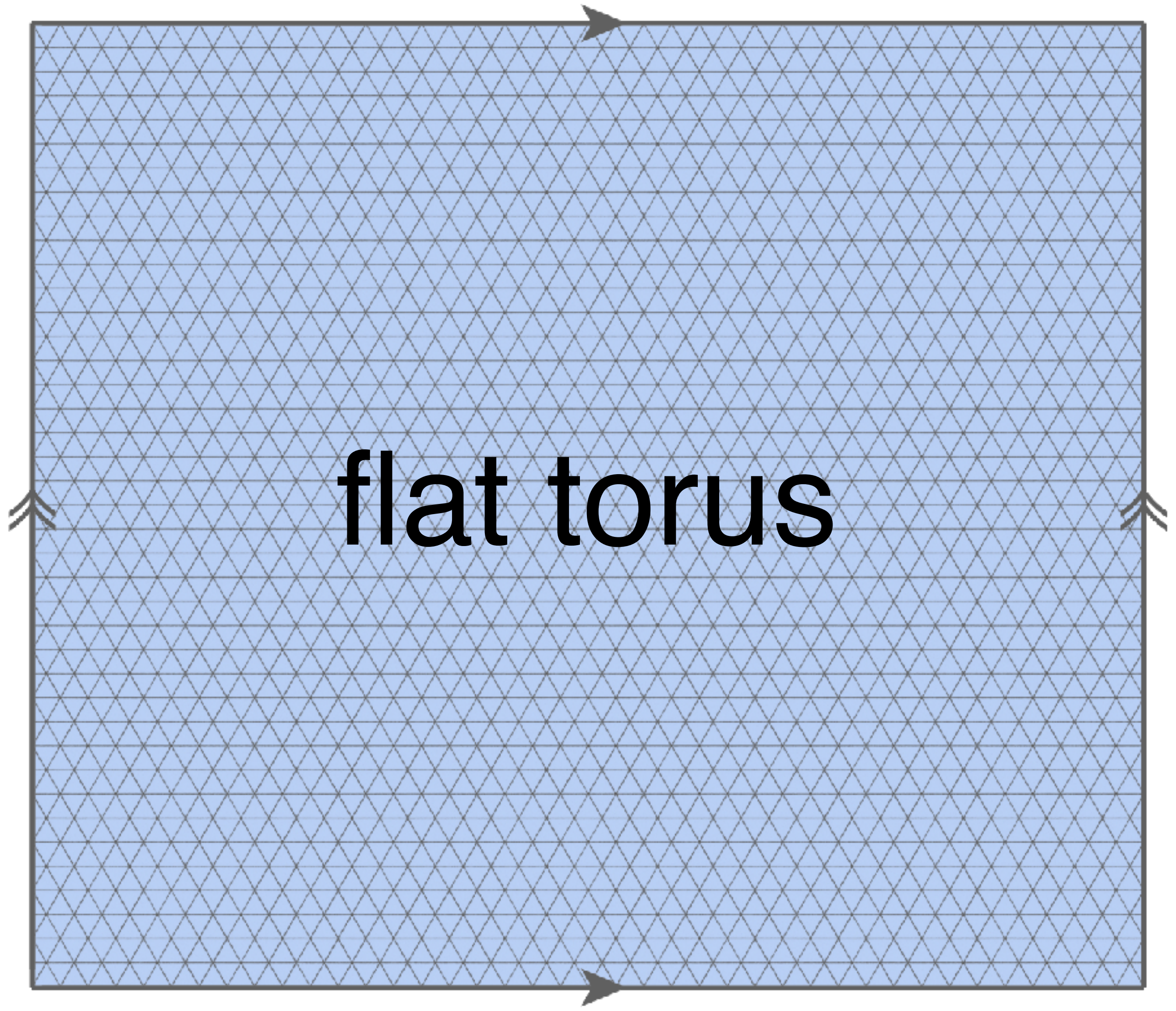


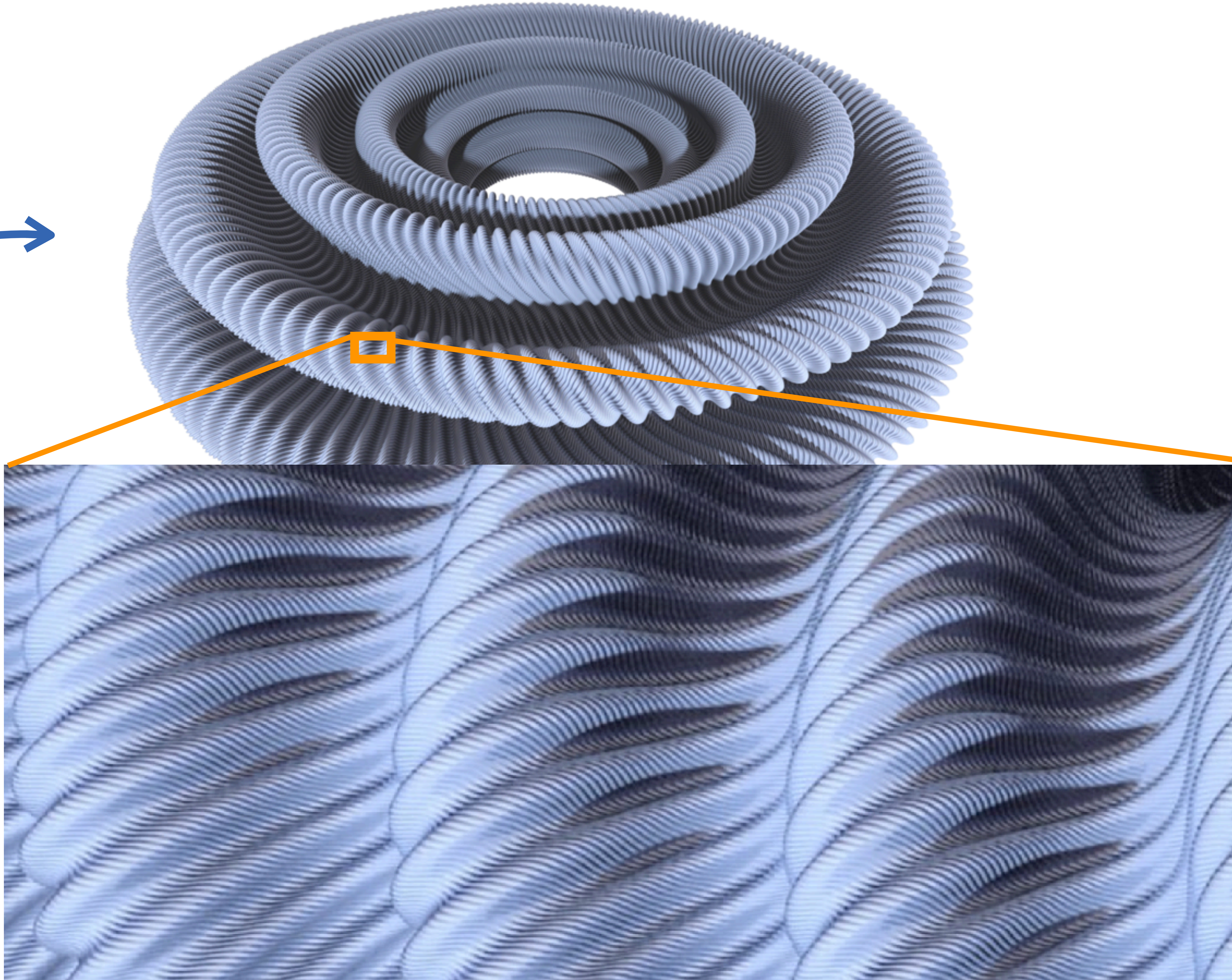


[Borrelli, Jabrane, Lazarus, Rohmer & Thibert 2012]

Flat torus

C^1 embedding



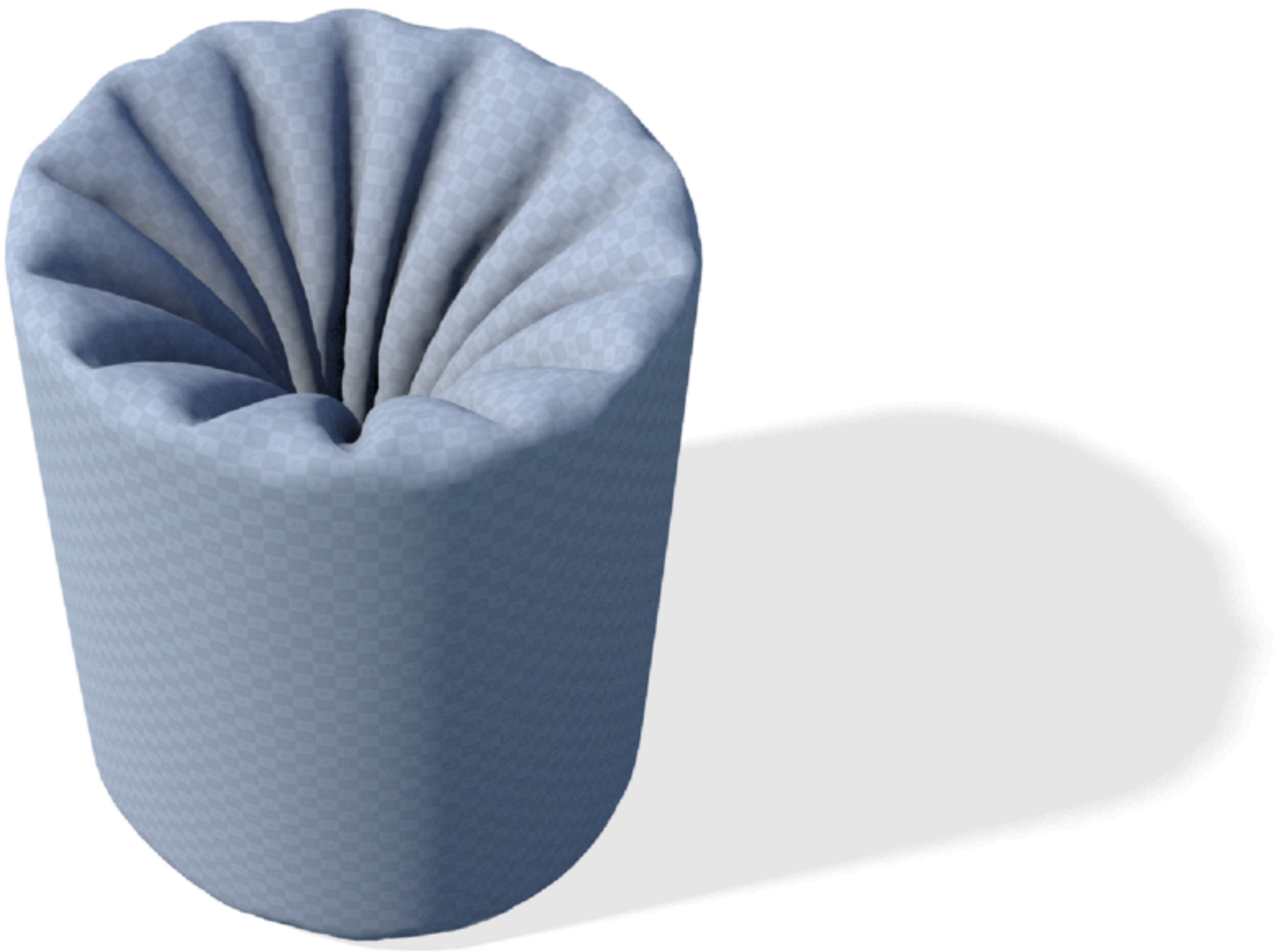


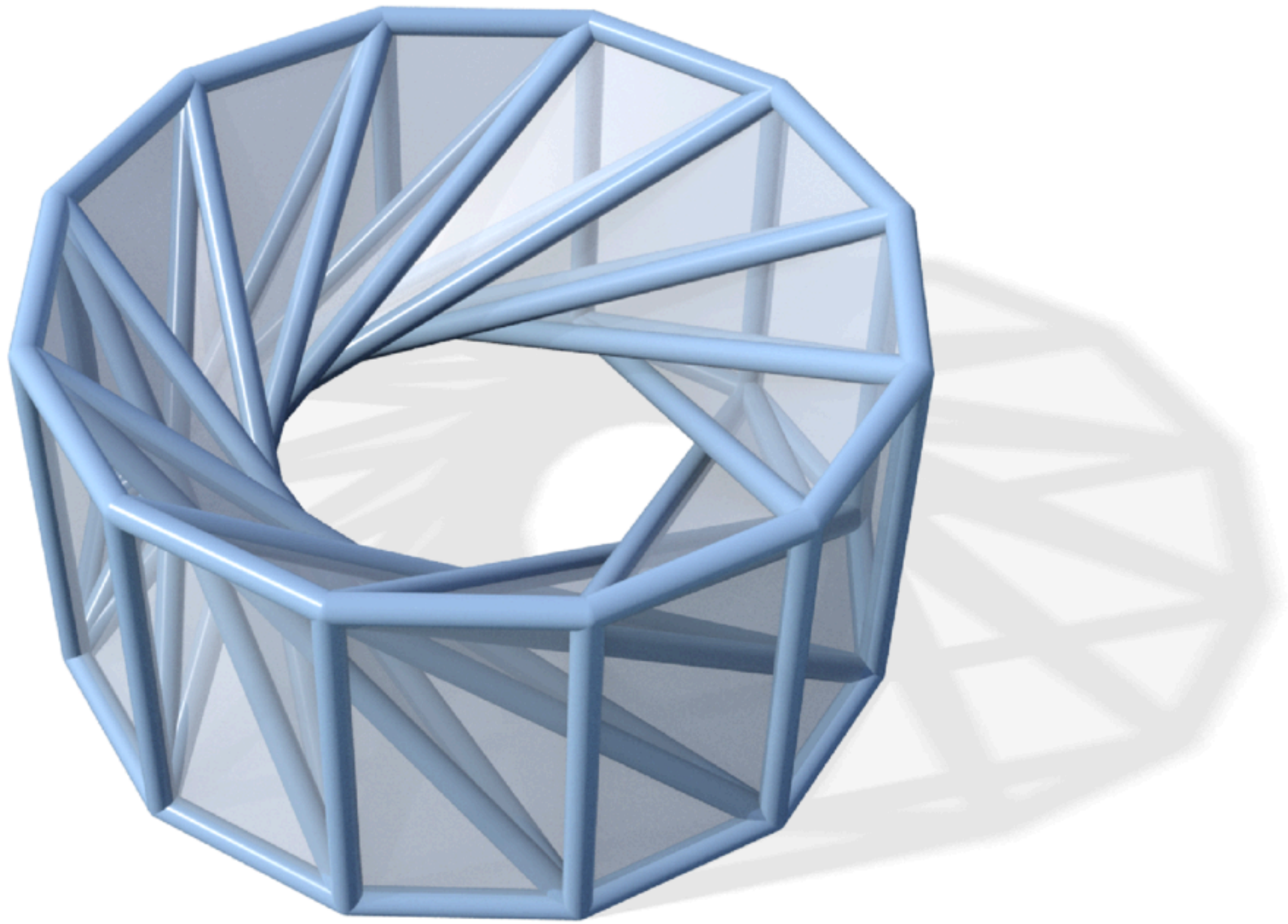
Flat torus

Piecewise smooth
C^0 embedding

Flat torus

Flat torus

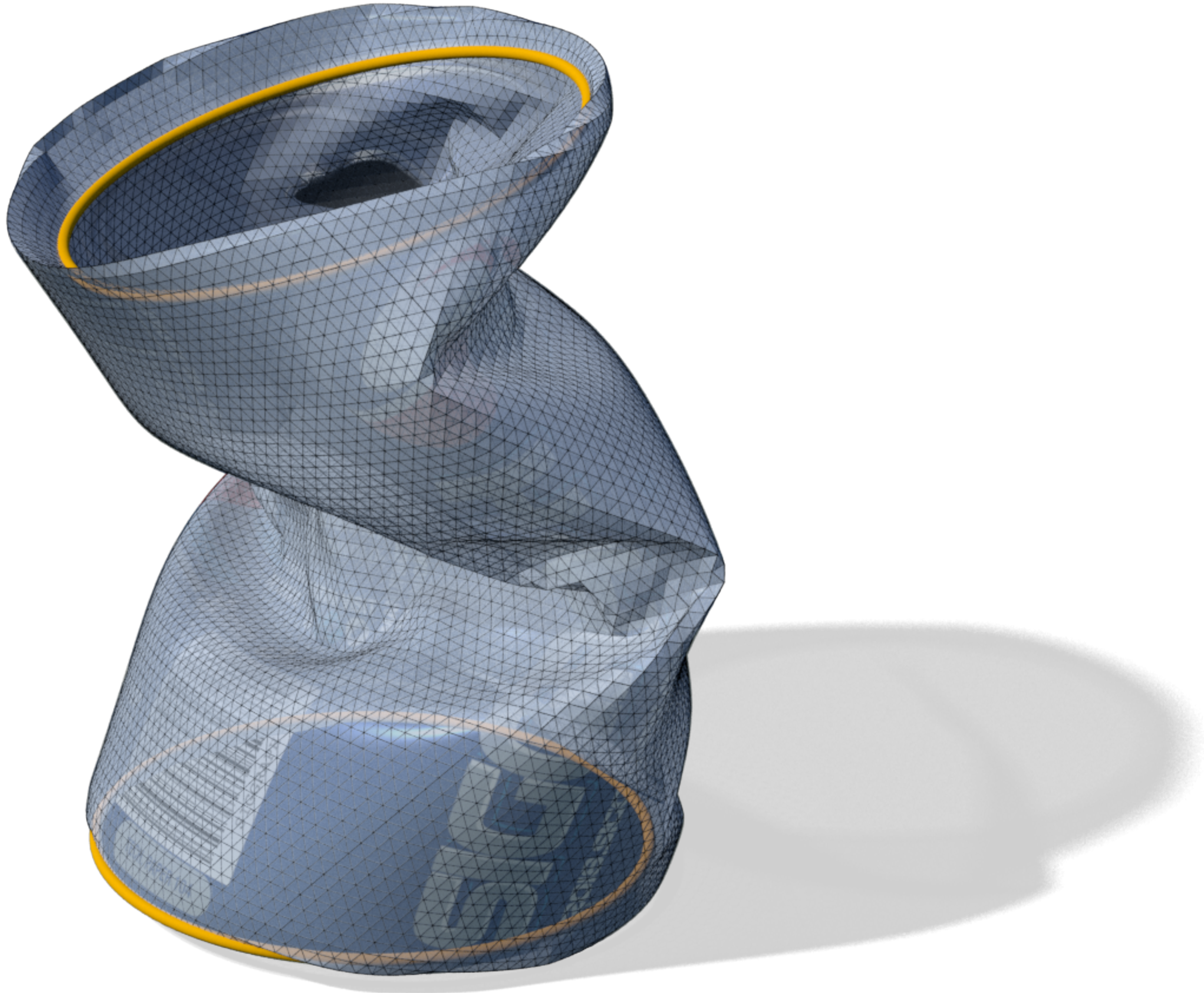




[H. Segerman 2015 *Shapeways*]
[R. Ferréol 2008 mathcurve.com]

Piecewise smooth embedding

Piecewise smooth embedding



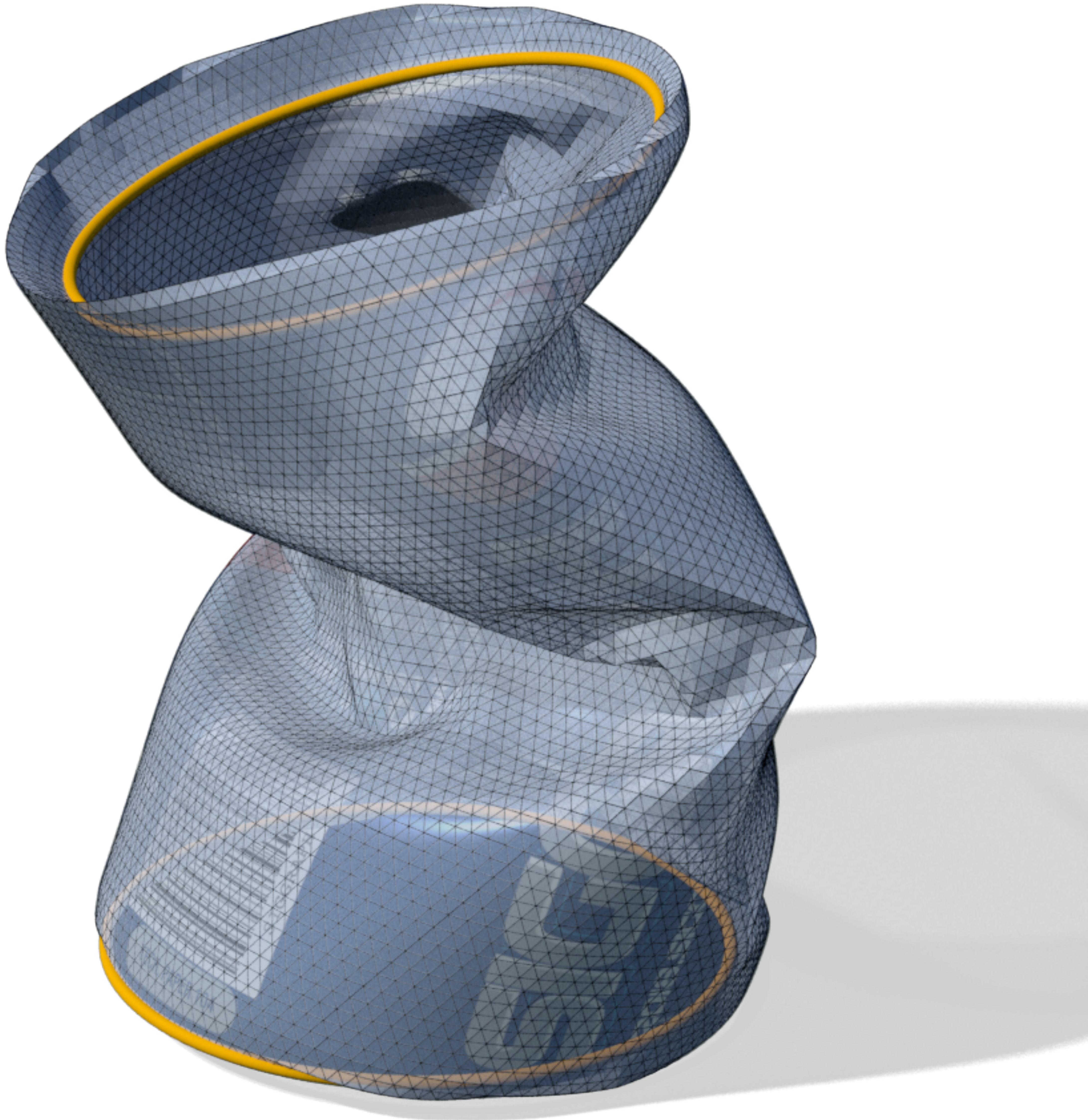
Piecewise smooth embedding

Microscopic scale

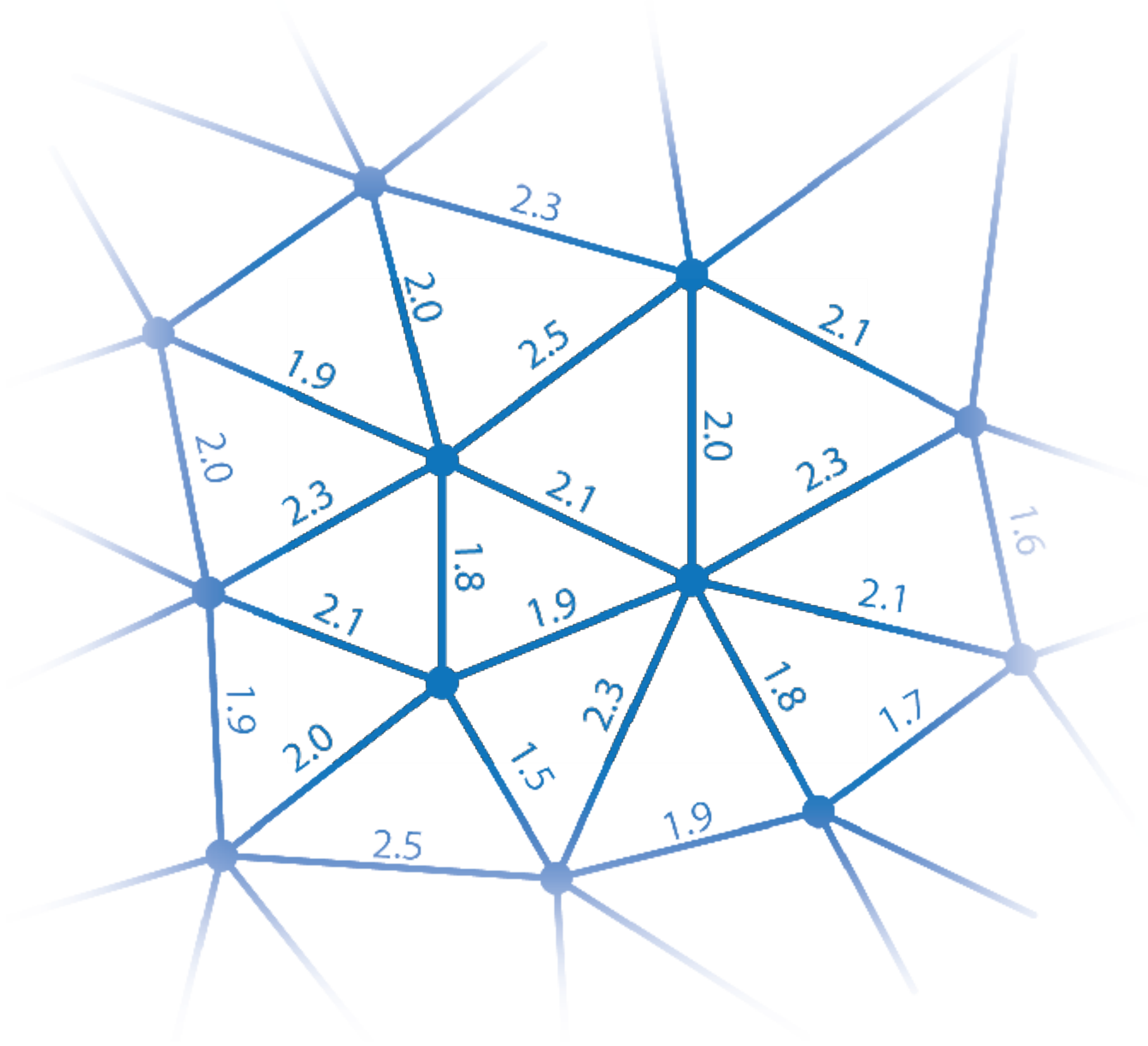
Isometry problem in
Euclidean plane.

Macroscopic scale

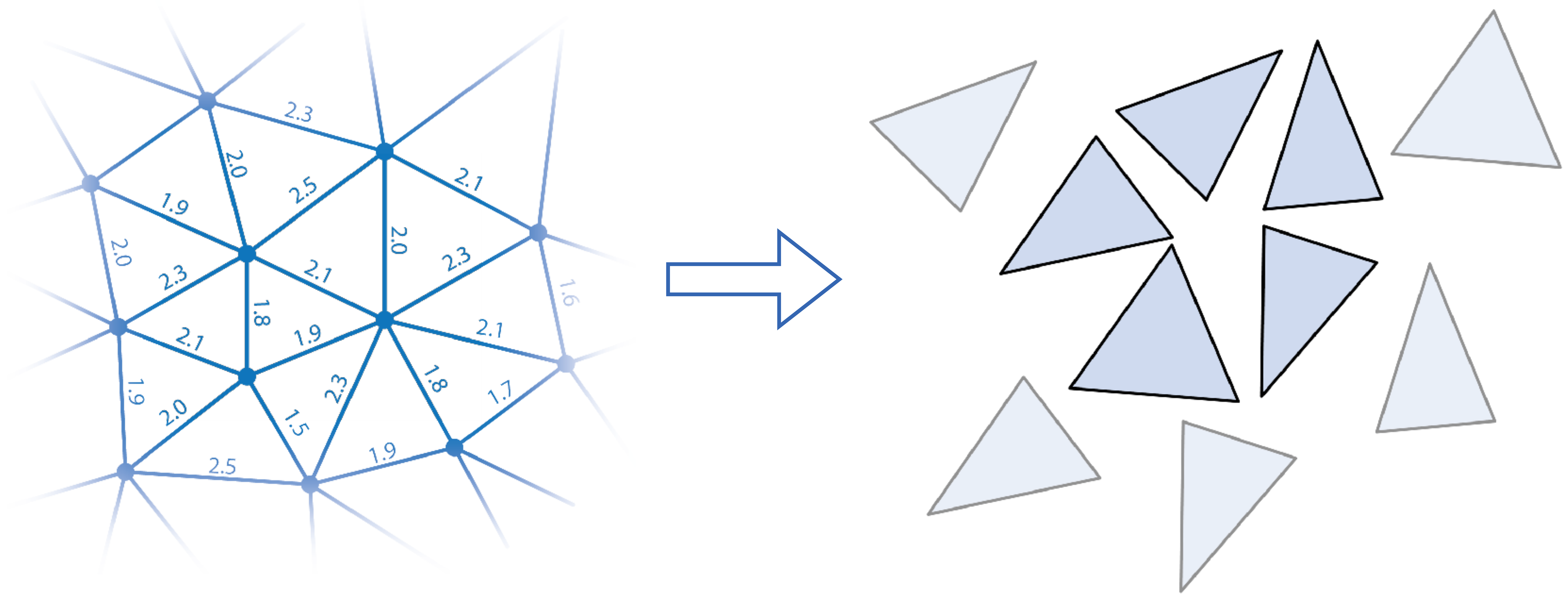
Gauge field theory.
Variational problem.



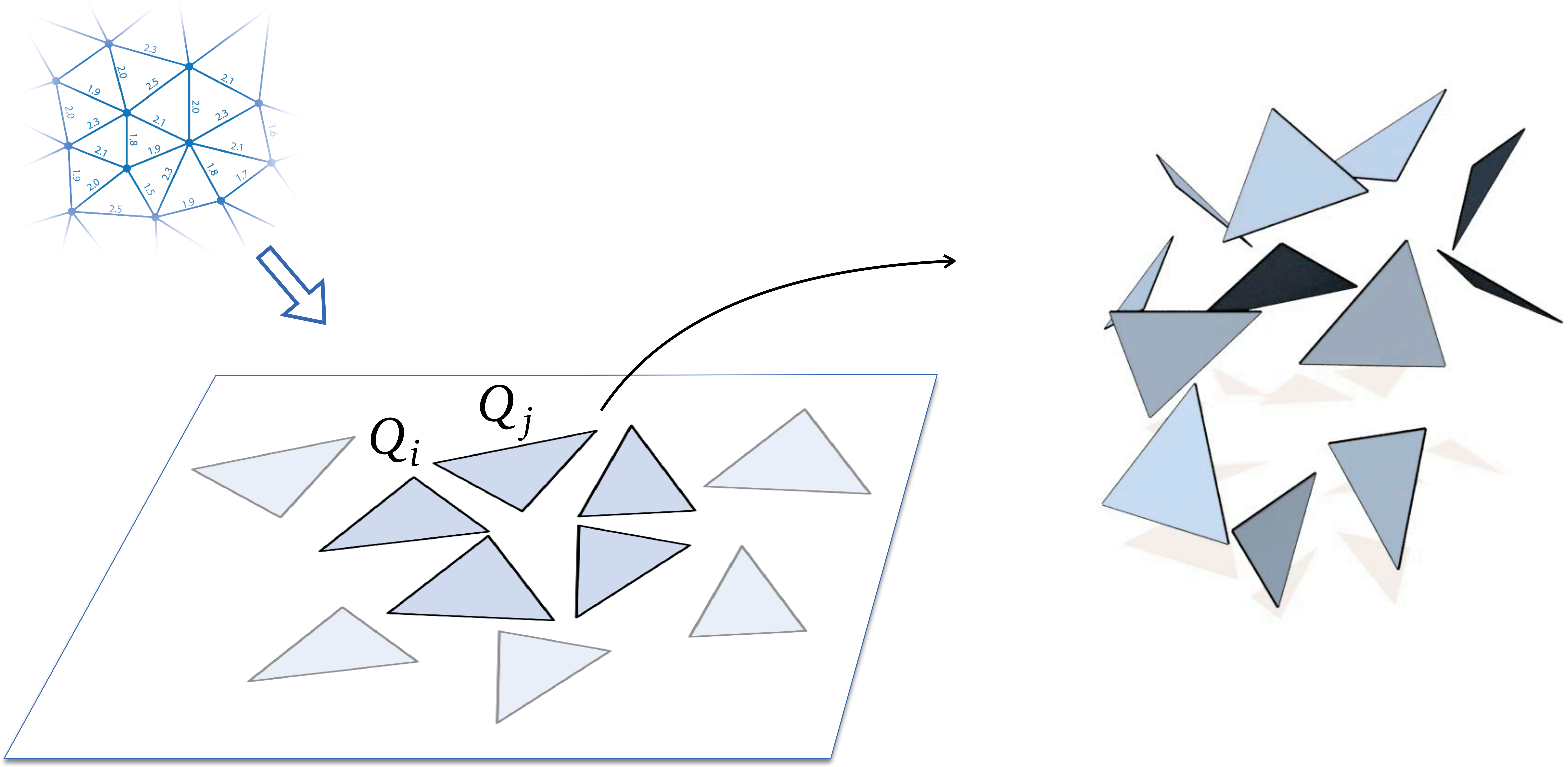
Microscopic level



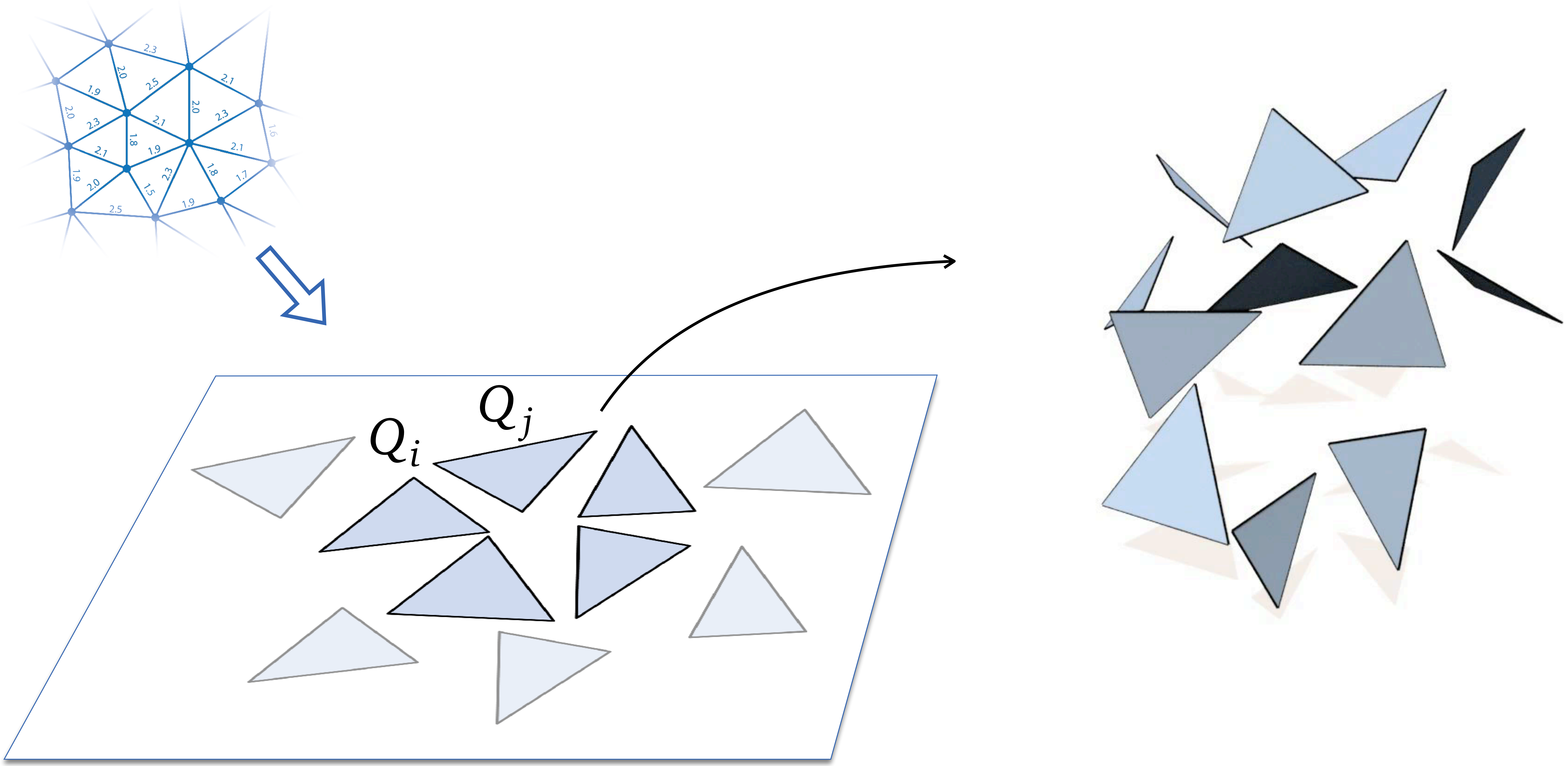
Microscopic level



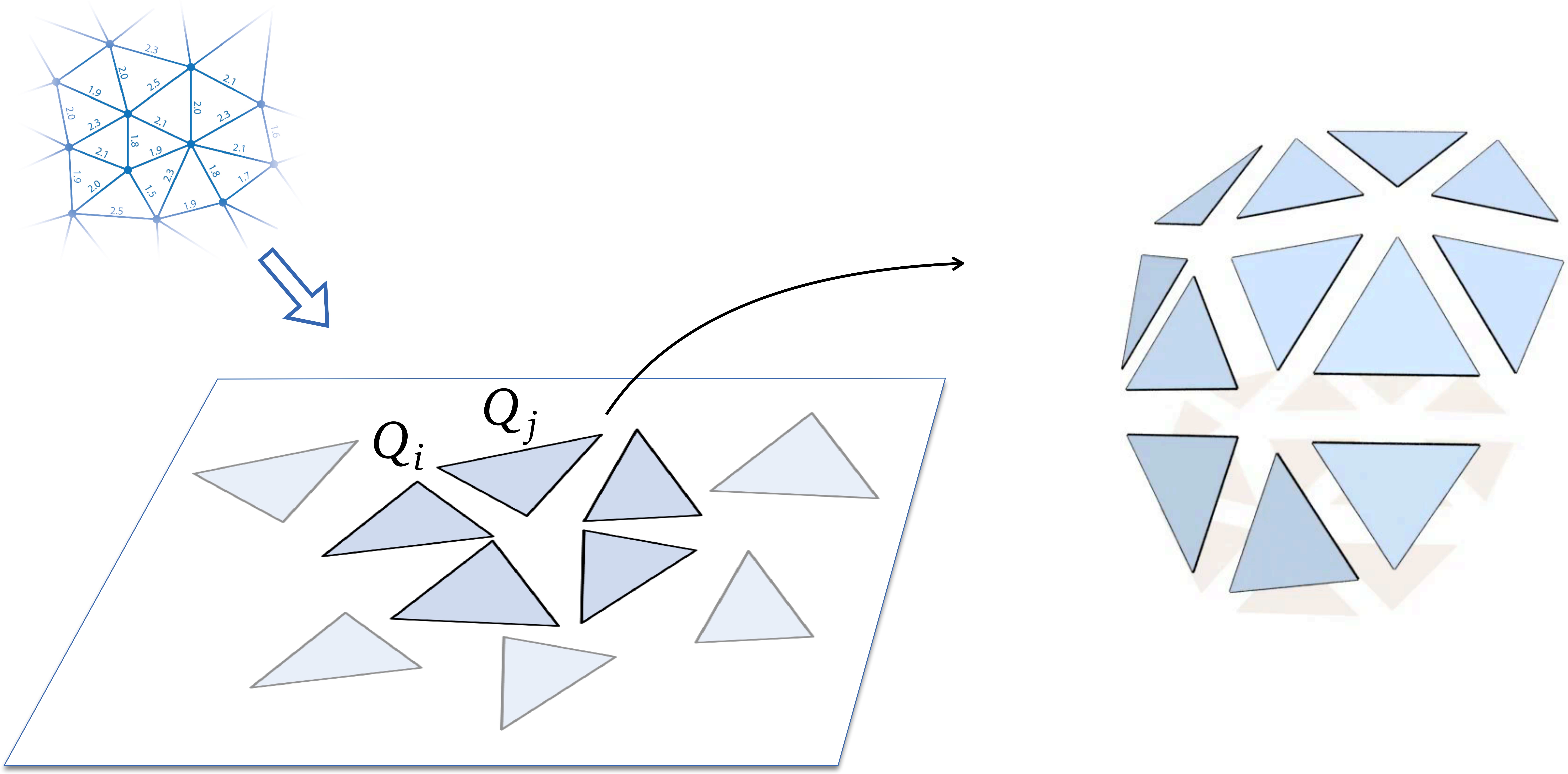
Global level — rotation field



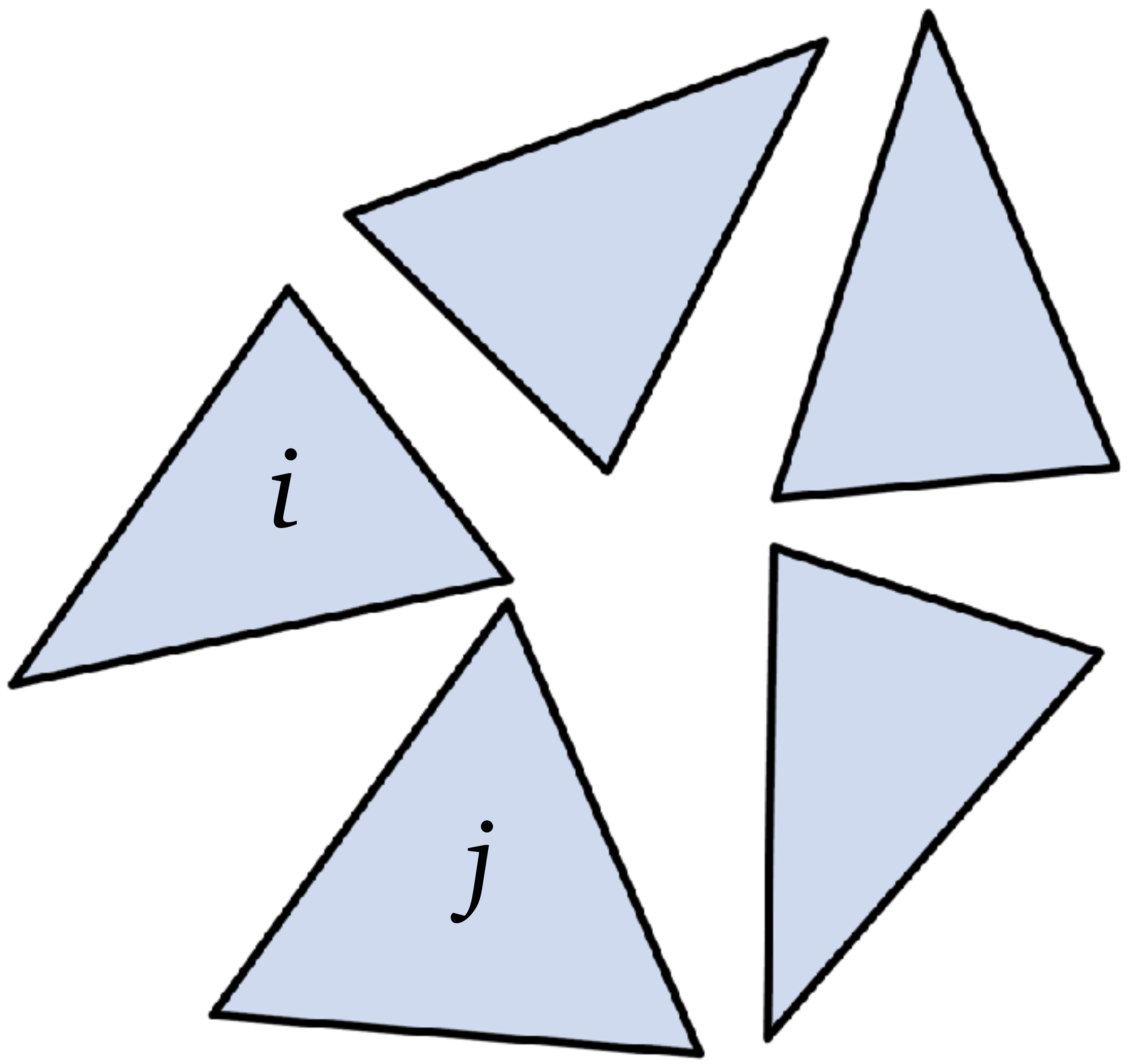
Global level — rotation field



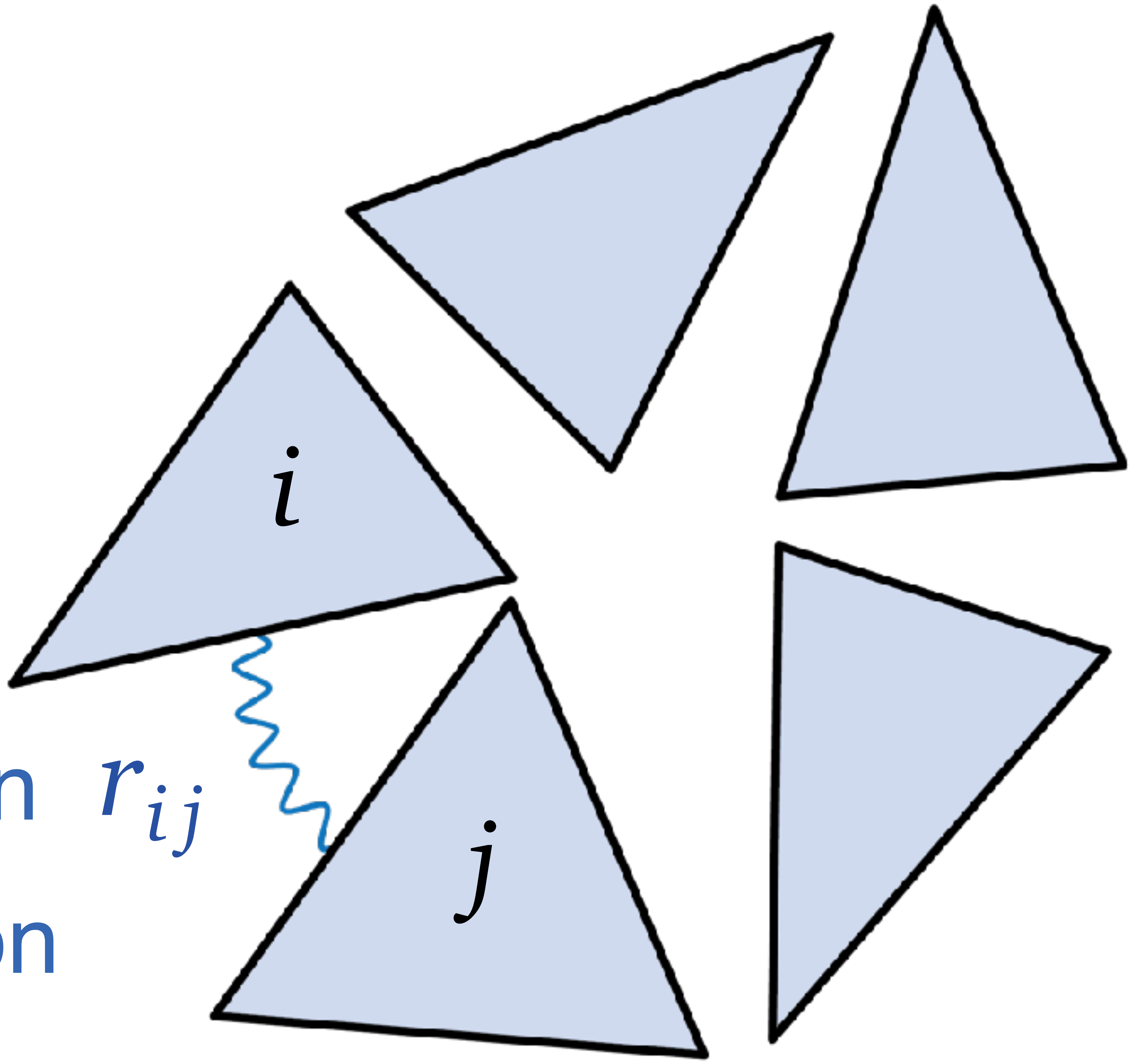
Global level — rotation field



Gauge theory



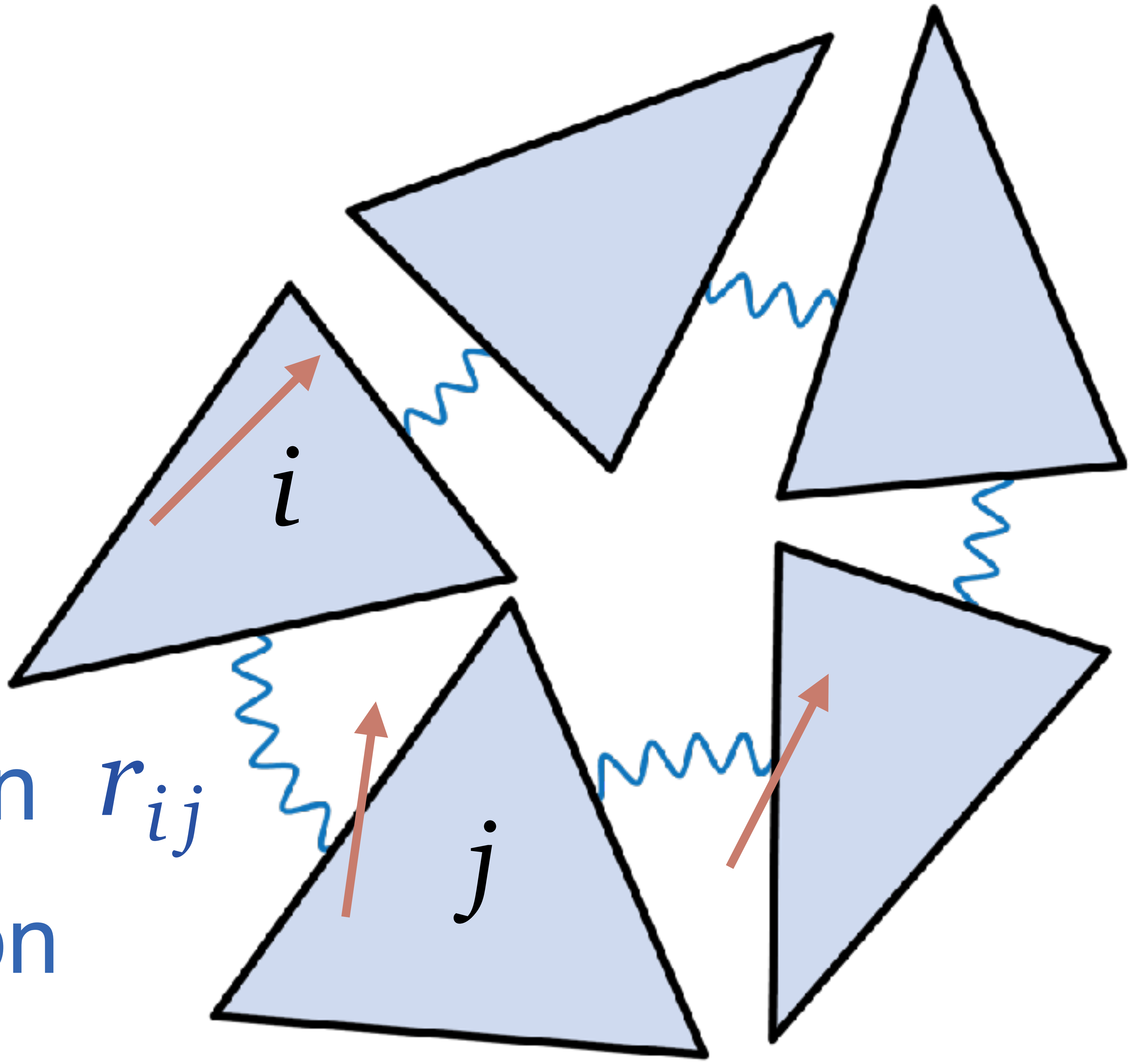
Gauge theory



Rotational connection r_{ij}

Levi-Civita connection

Gauge theory



Rotational connection r_{ij}

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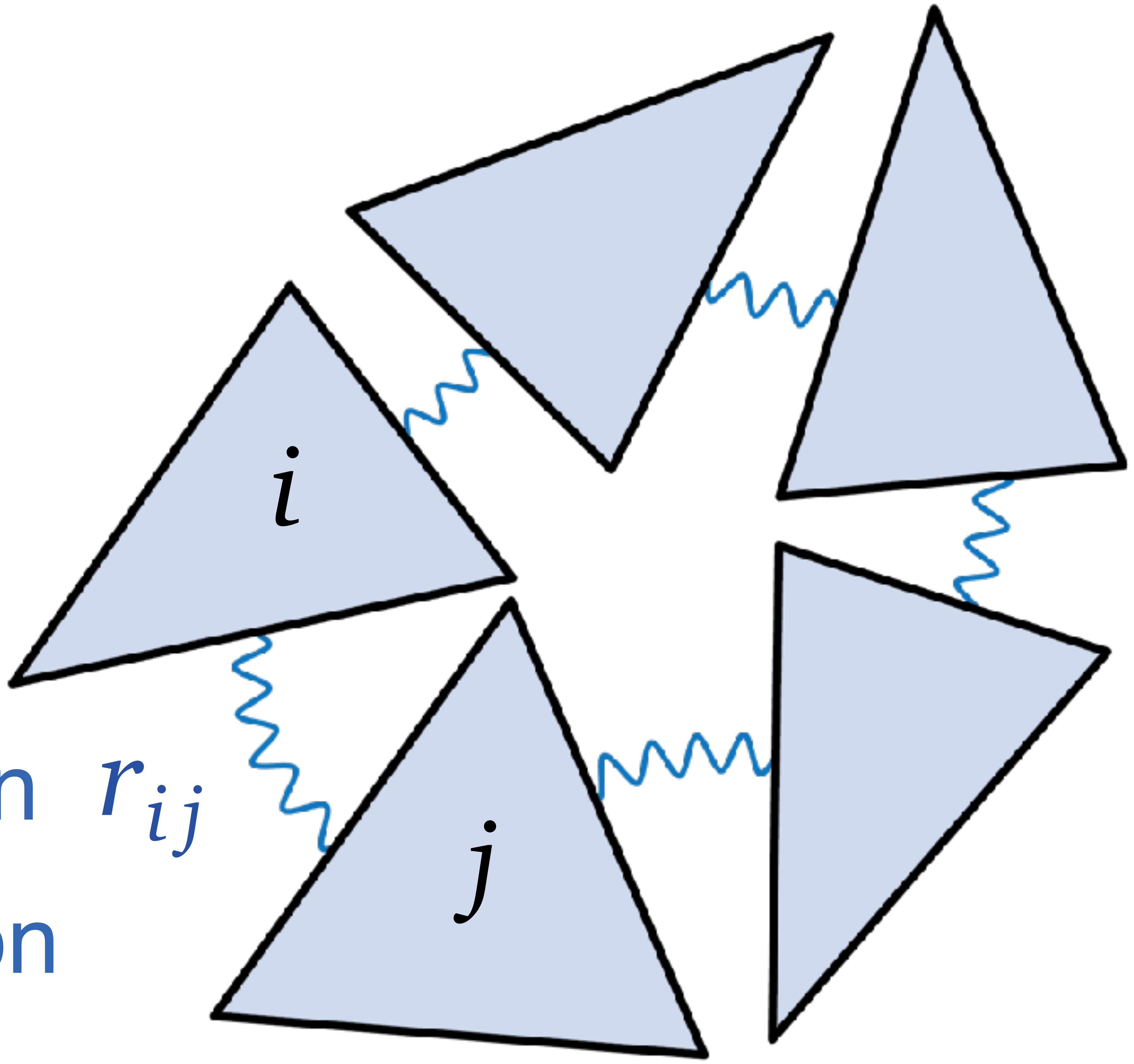
Gauge theory



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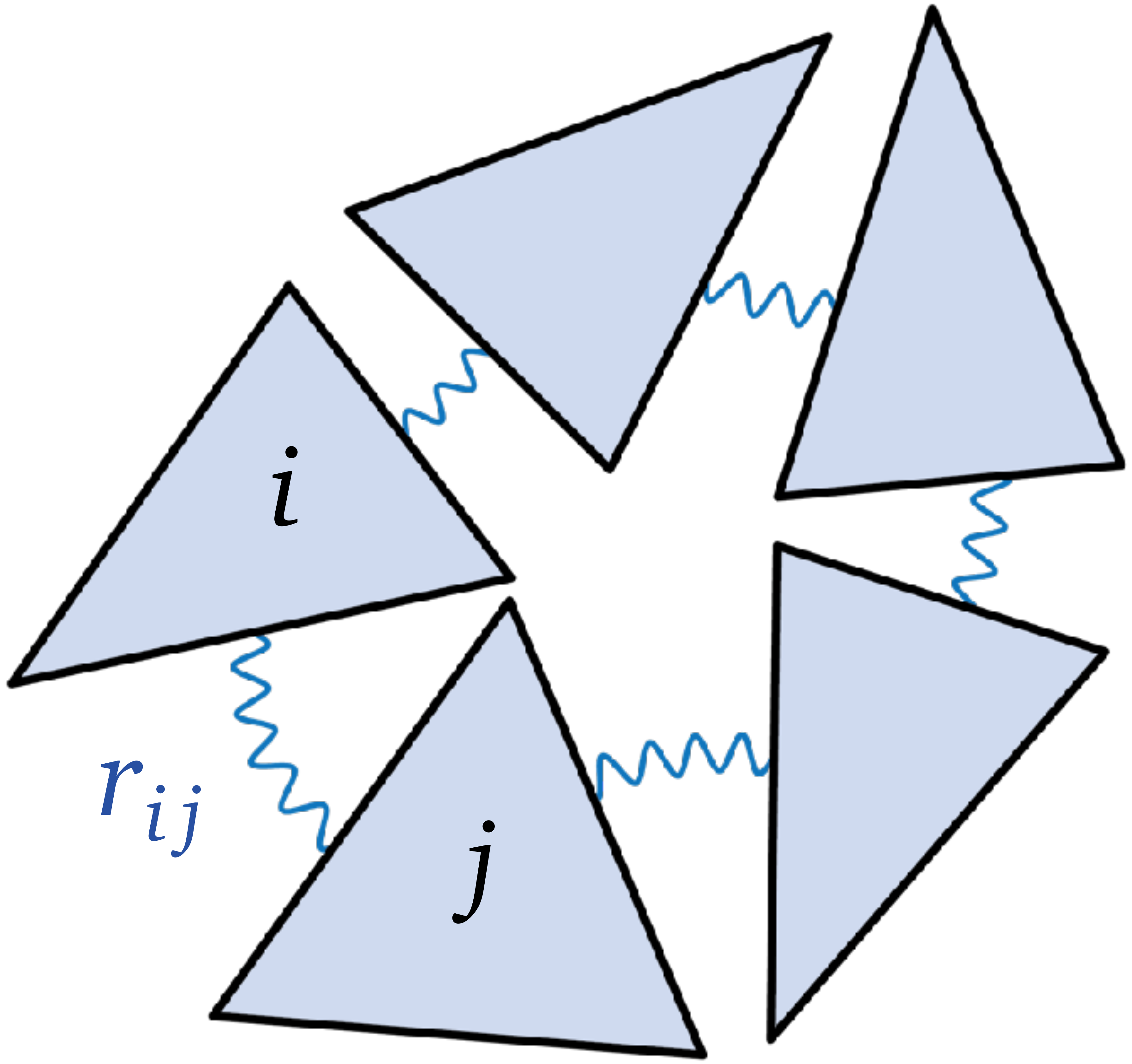
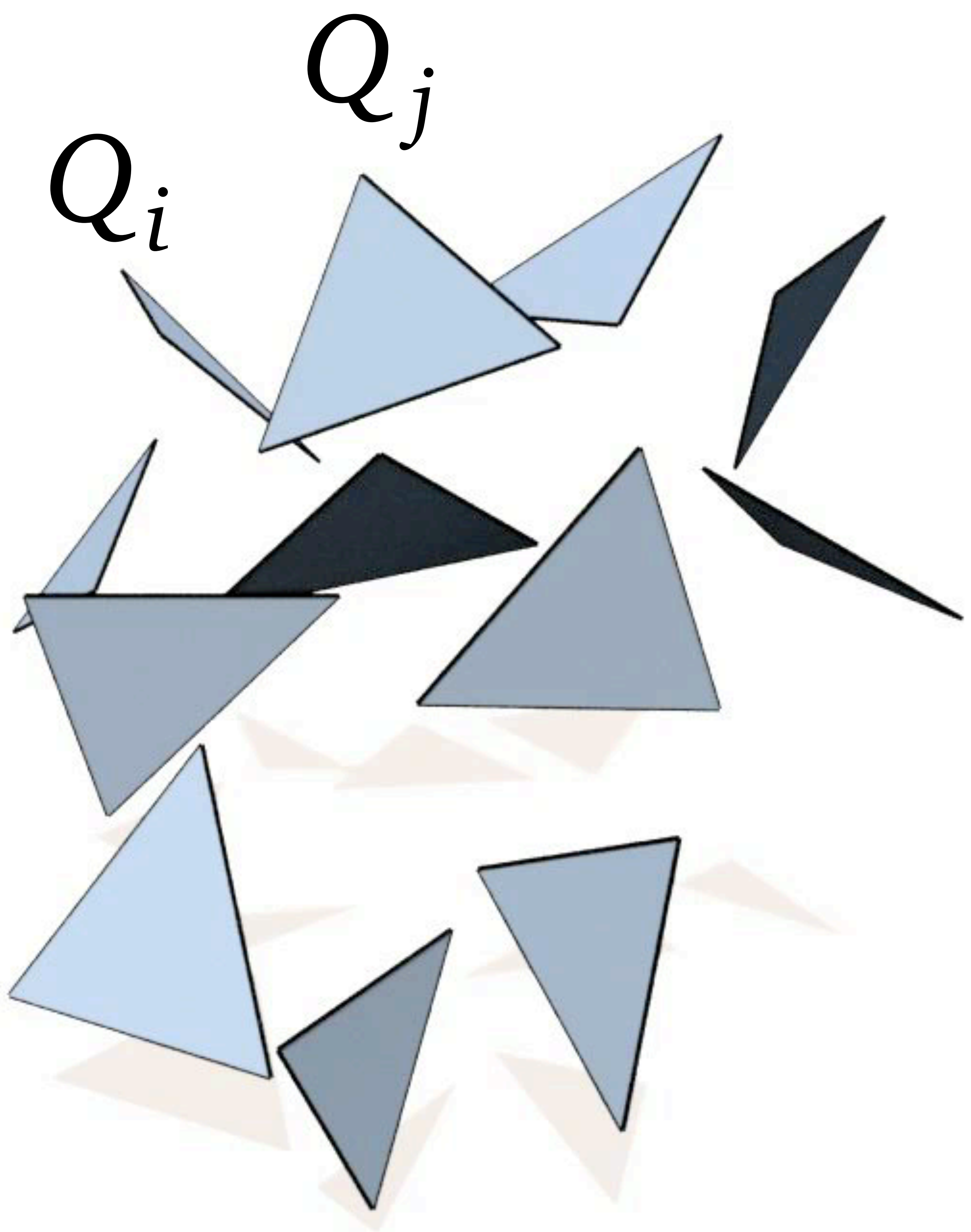
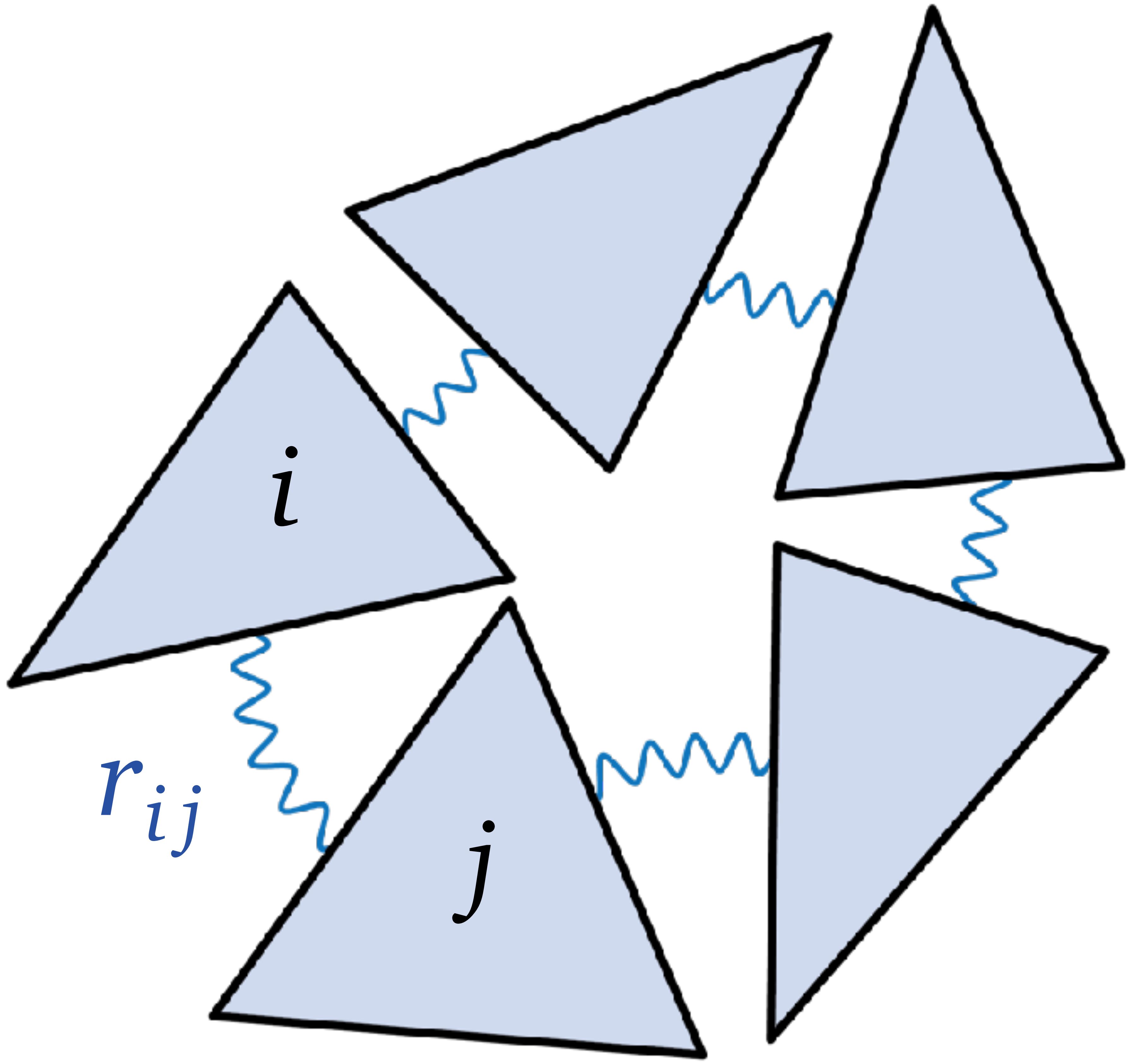




Gauge theory

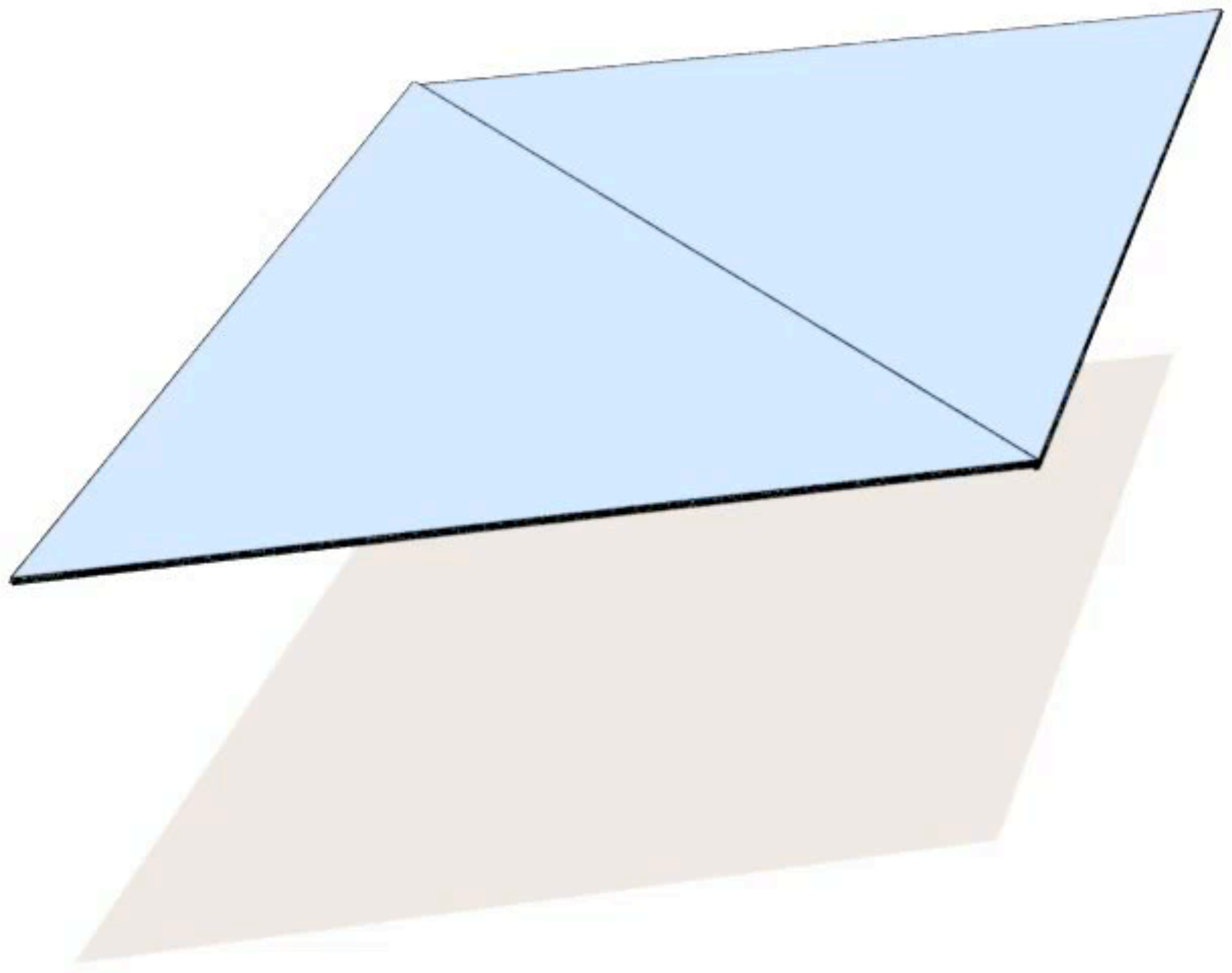


$$\left|\boxed{Q_j - Q_i \circ r_{ij}}\right|$$

connection derivative

Gauge theory

$Q_j - Q_i \circ r_{ij}$ contains 3 modes



Gauge theory

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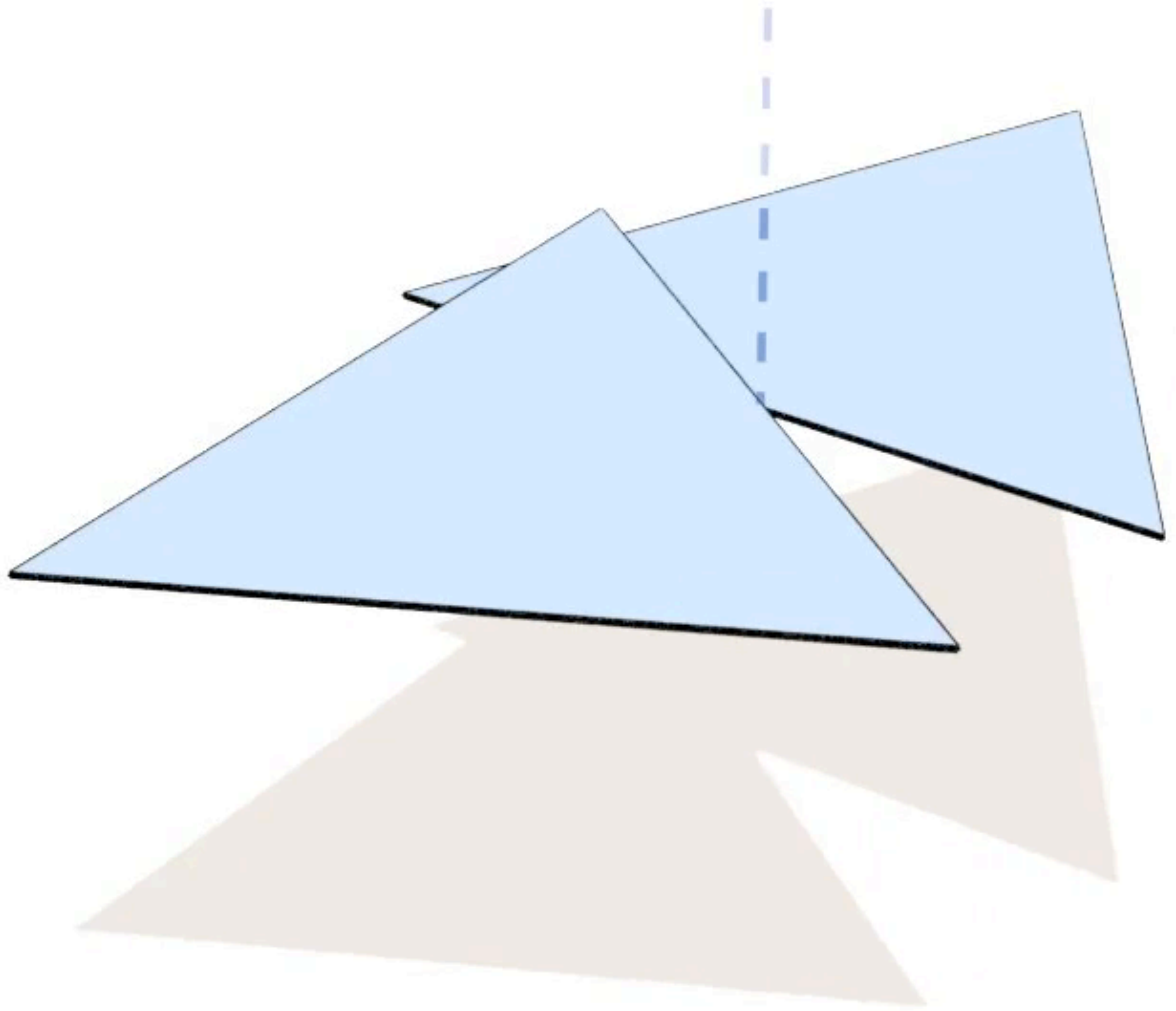

Gauge theory

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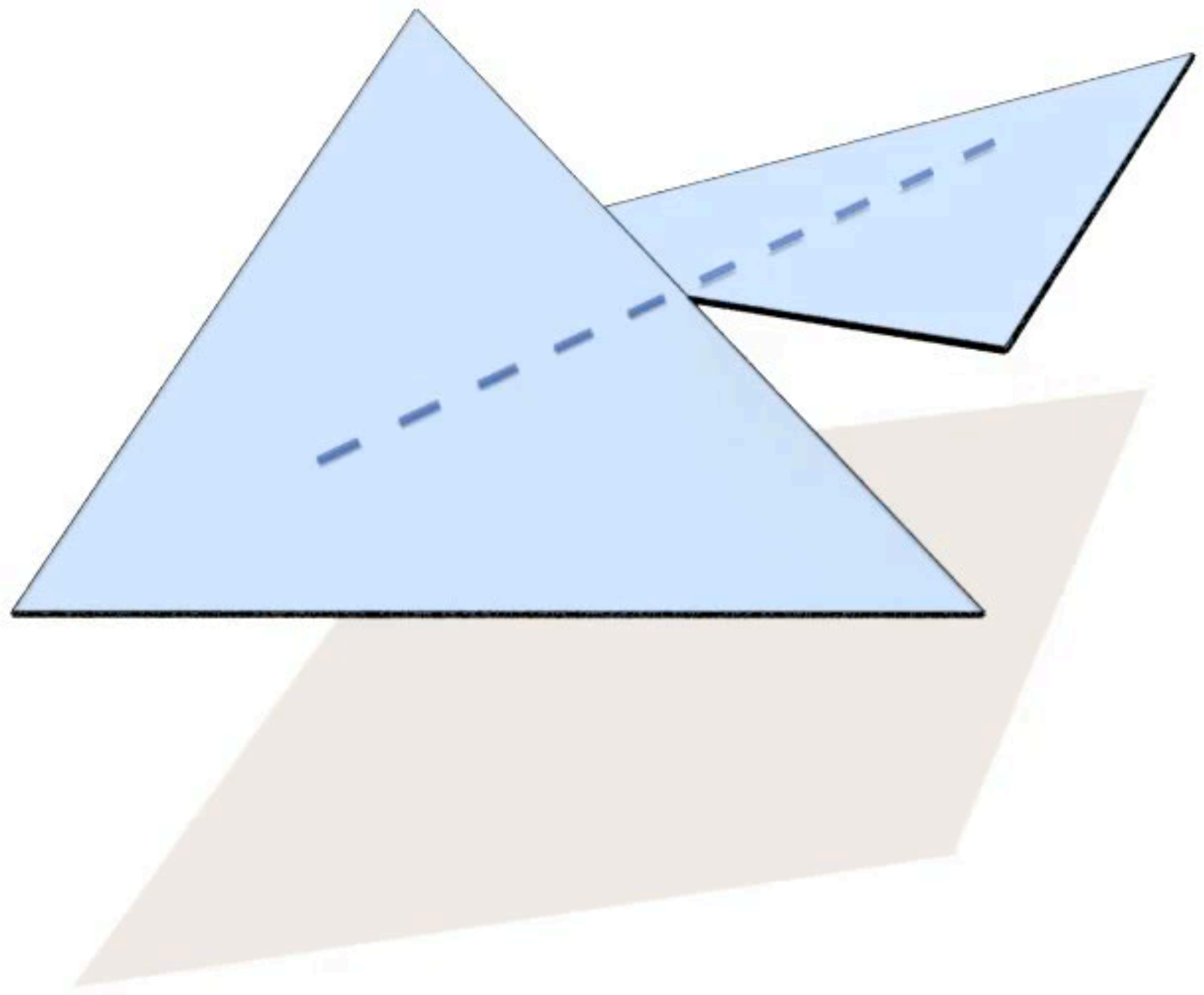

Gauge theory

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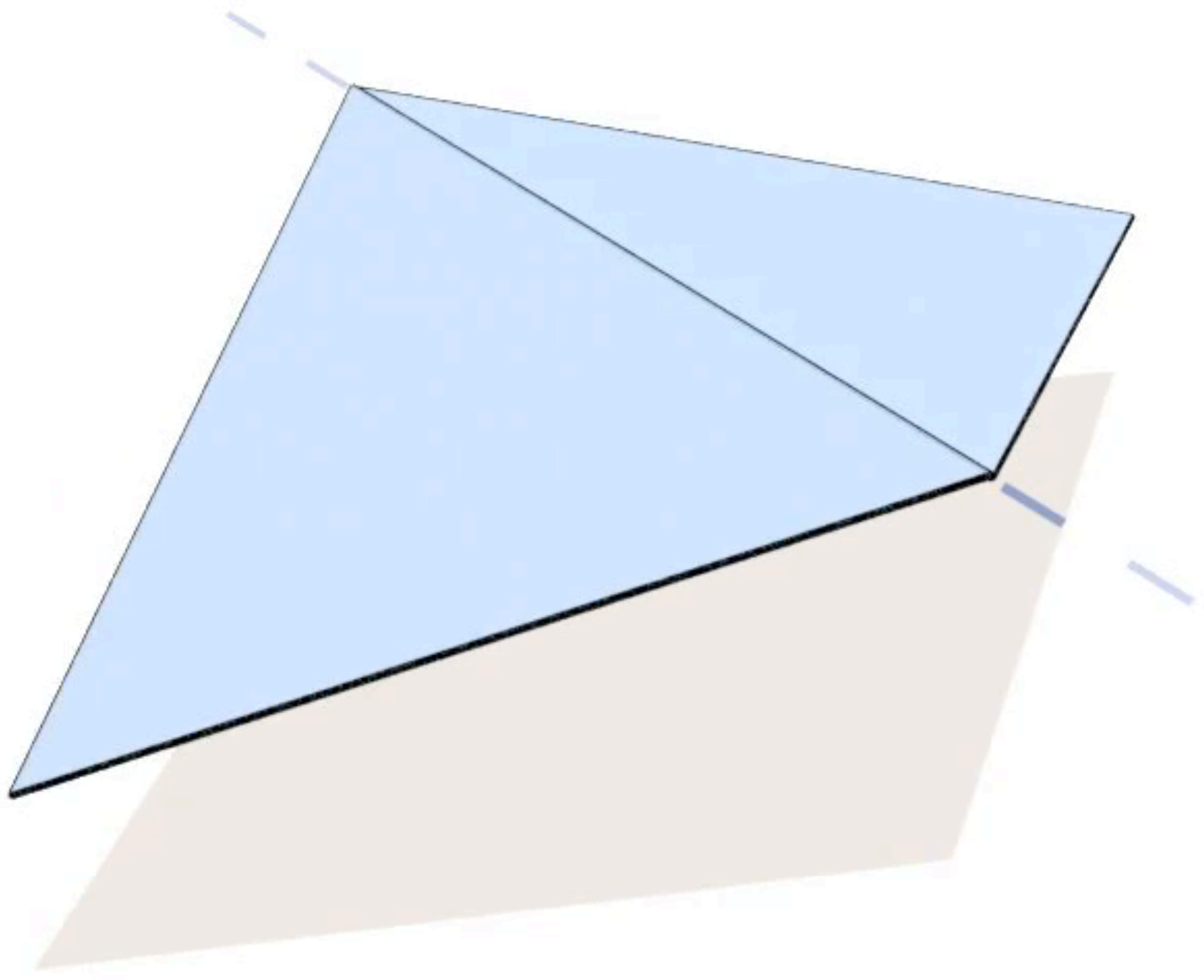
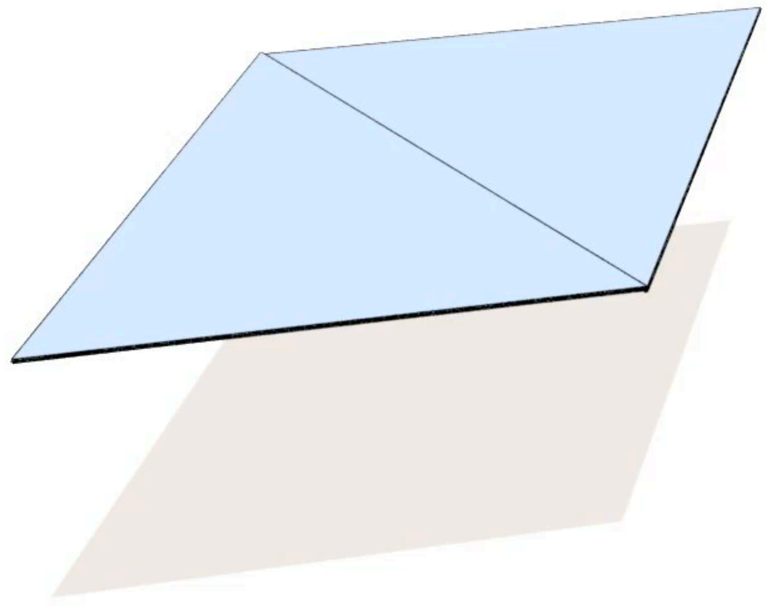
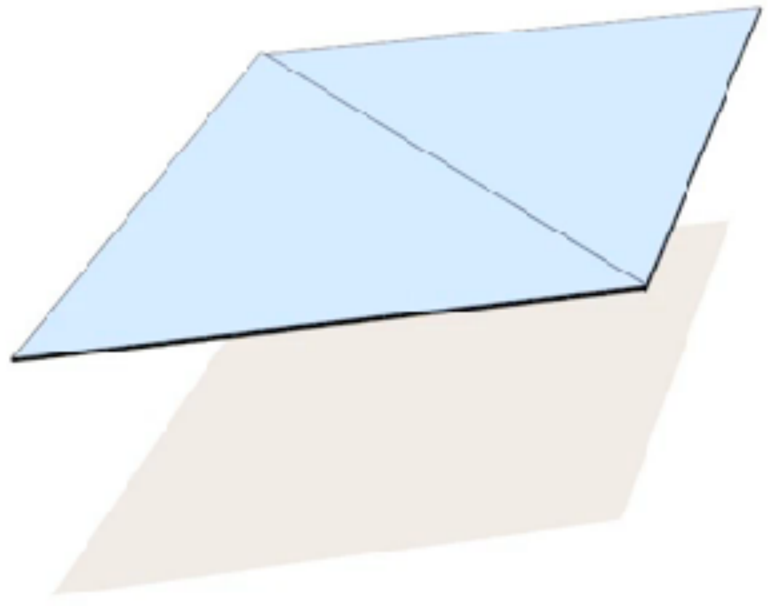
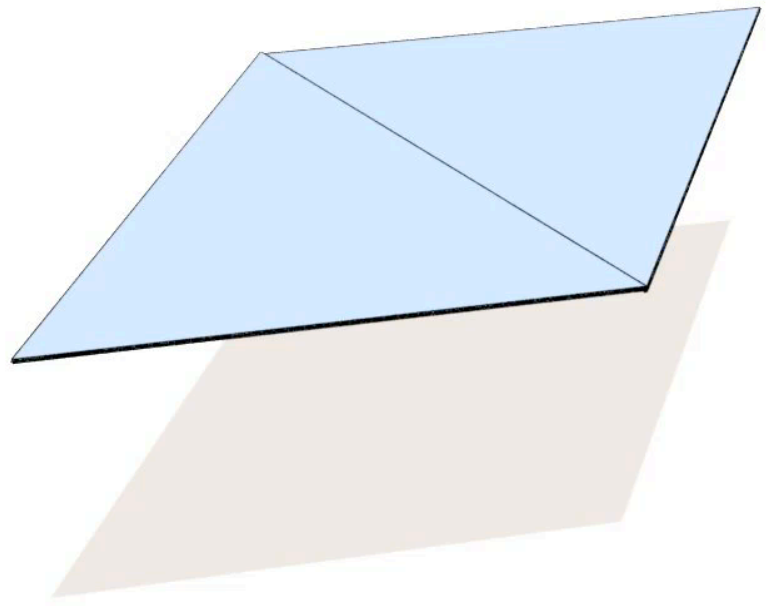


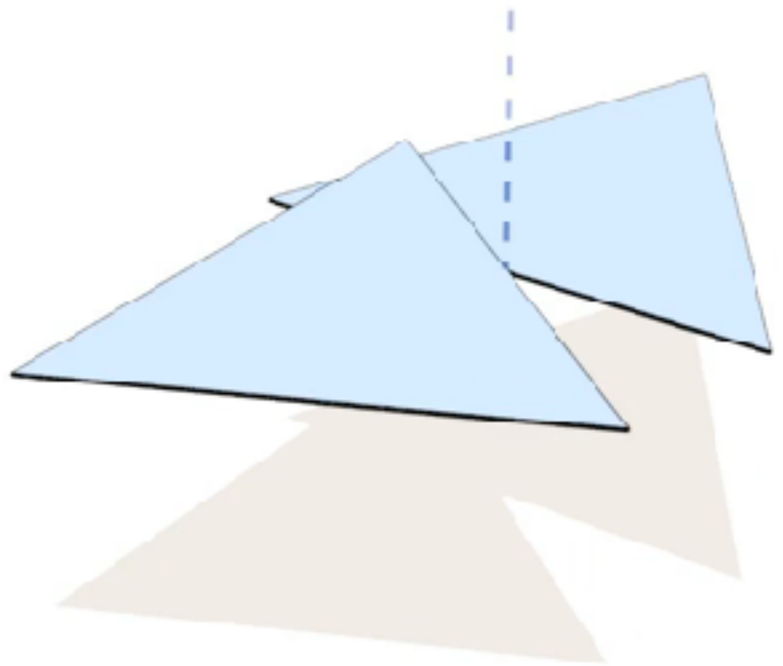
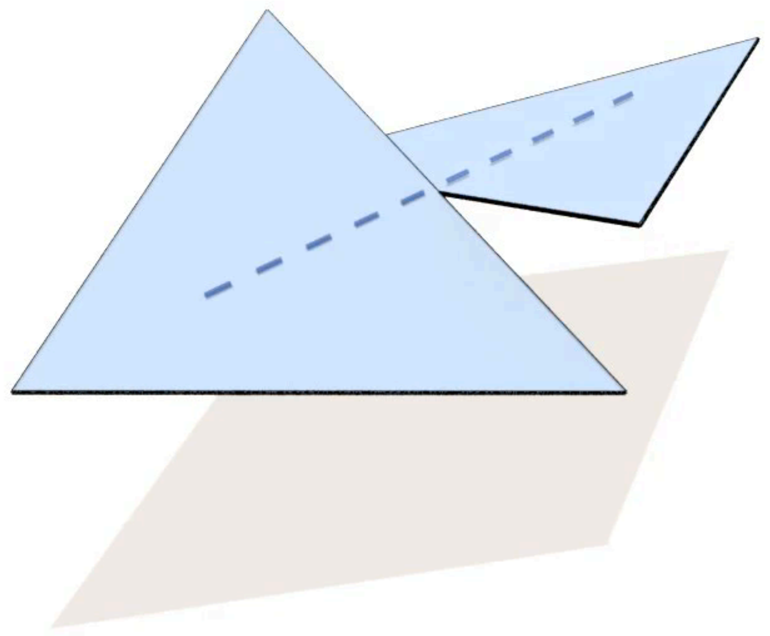
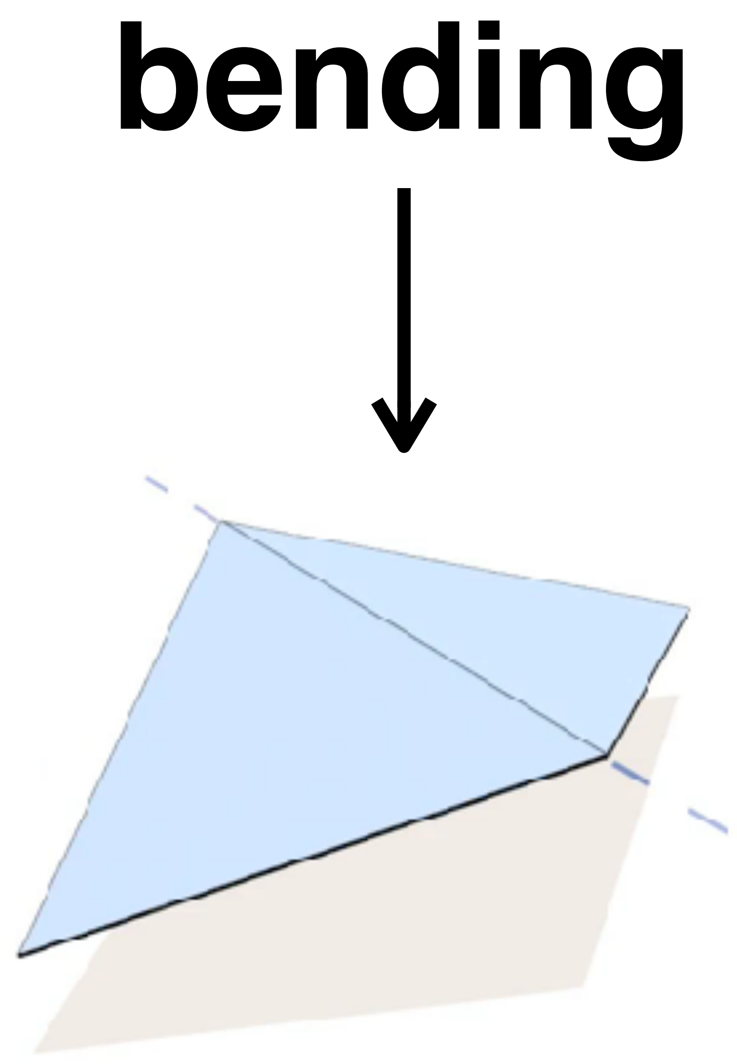
Gauge theory

$$Q_j - Q_i \circ r_{ij}$$

Gauge theory

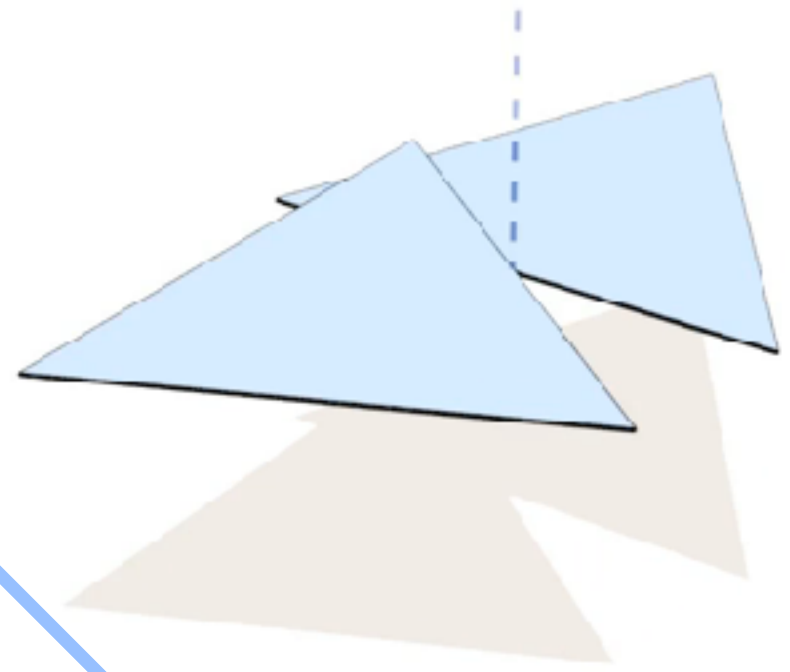
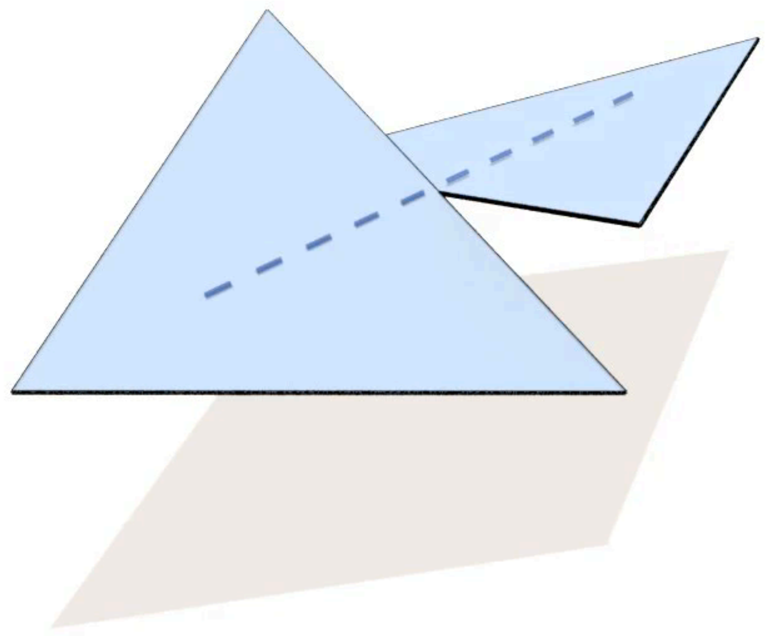
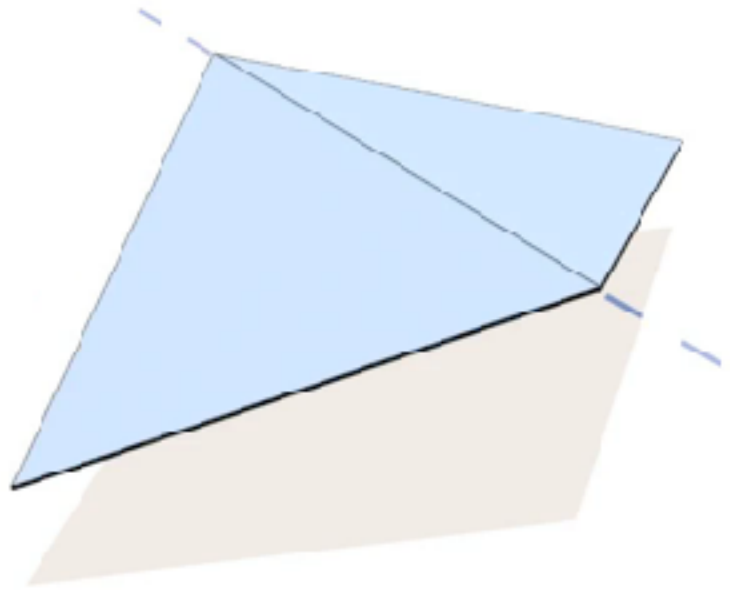
$Q_j - Q_i \circ r_{ij} \quad =$  $+$  $+$ 

Gauge theory

$Q_j - Q_i \circ r_{ij} \quad = $  $+$  $+$ 

Gauge theory

Anisotropic norm

$\left|Q_j - Q_i \circ r_{ij}\right|^2_\epsilon = \boxed{\epsilon_1}\Big|$  $\Big|^2 + \boxed{\epsilon_2}\Big|$  $\Big|^2 + \boxed{\epsilon_3}\Big|$  $\Big|^2$

fidelity

regularization

Energy functional

Dirichlet energy

$$\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_{\epsilon}^{2}$$

Energy functional

Dirichlet energy

$$\sum_{\text{all edges}} \left| Q_j - Q_i \circ \boxed{r_{ij}} \right|_\epsilon^2$$

connection

Energy functional

Ginzburg—Landau energy

$$\sum_{\text{all edges}} \left| \boxed{Q_j} - \boxed{Q_i} \circ \boxed{r_{ij}} \right|_\epsilon^2$$

fermions

gauge (boson) field

Energy functional

Ginzburg—Landau energy

$$\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|^2_{\boxed{\epsilon}}$$

anisotropic norm

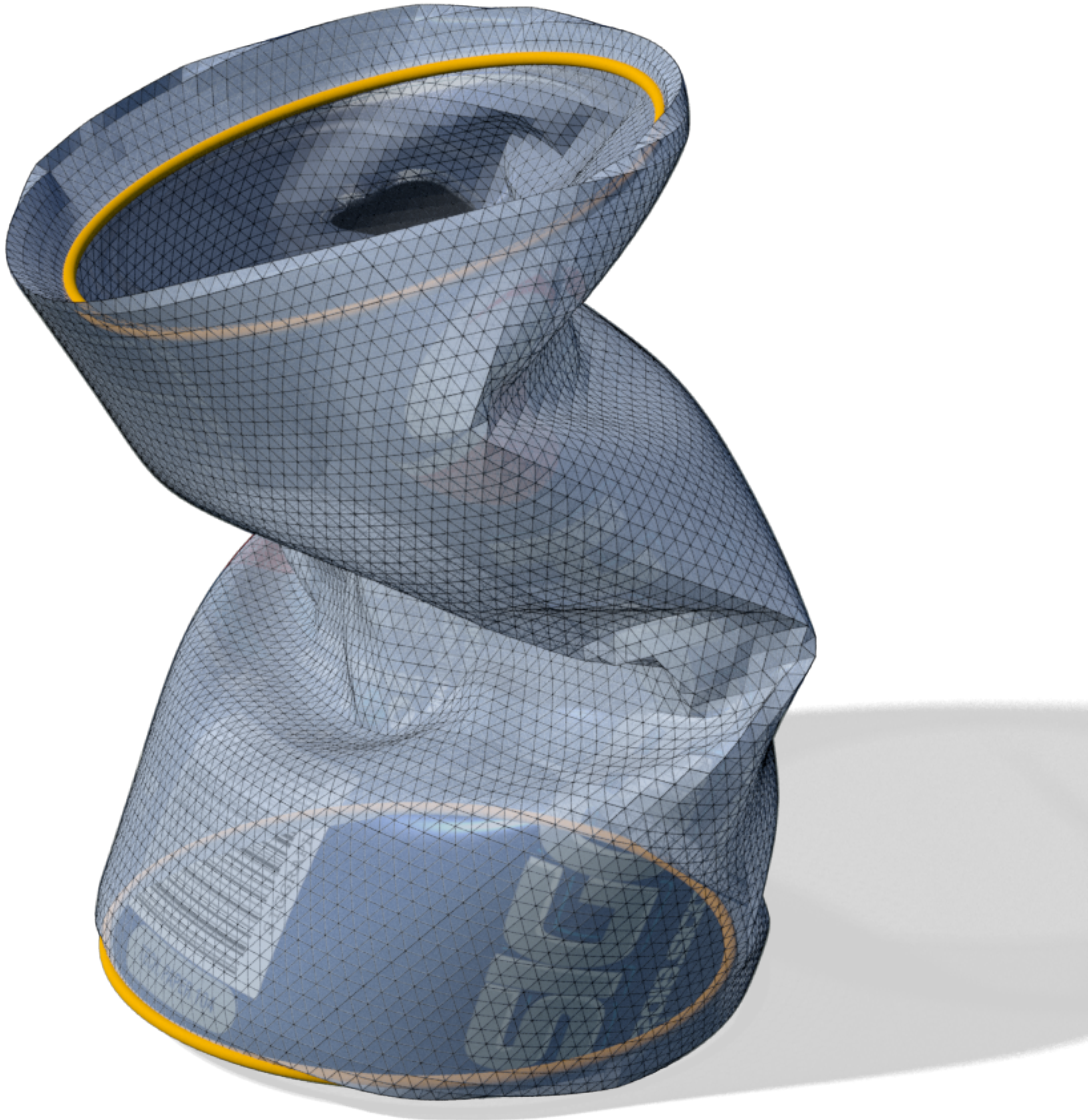
Emergent surface

Microscopic scale

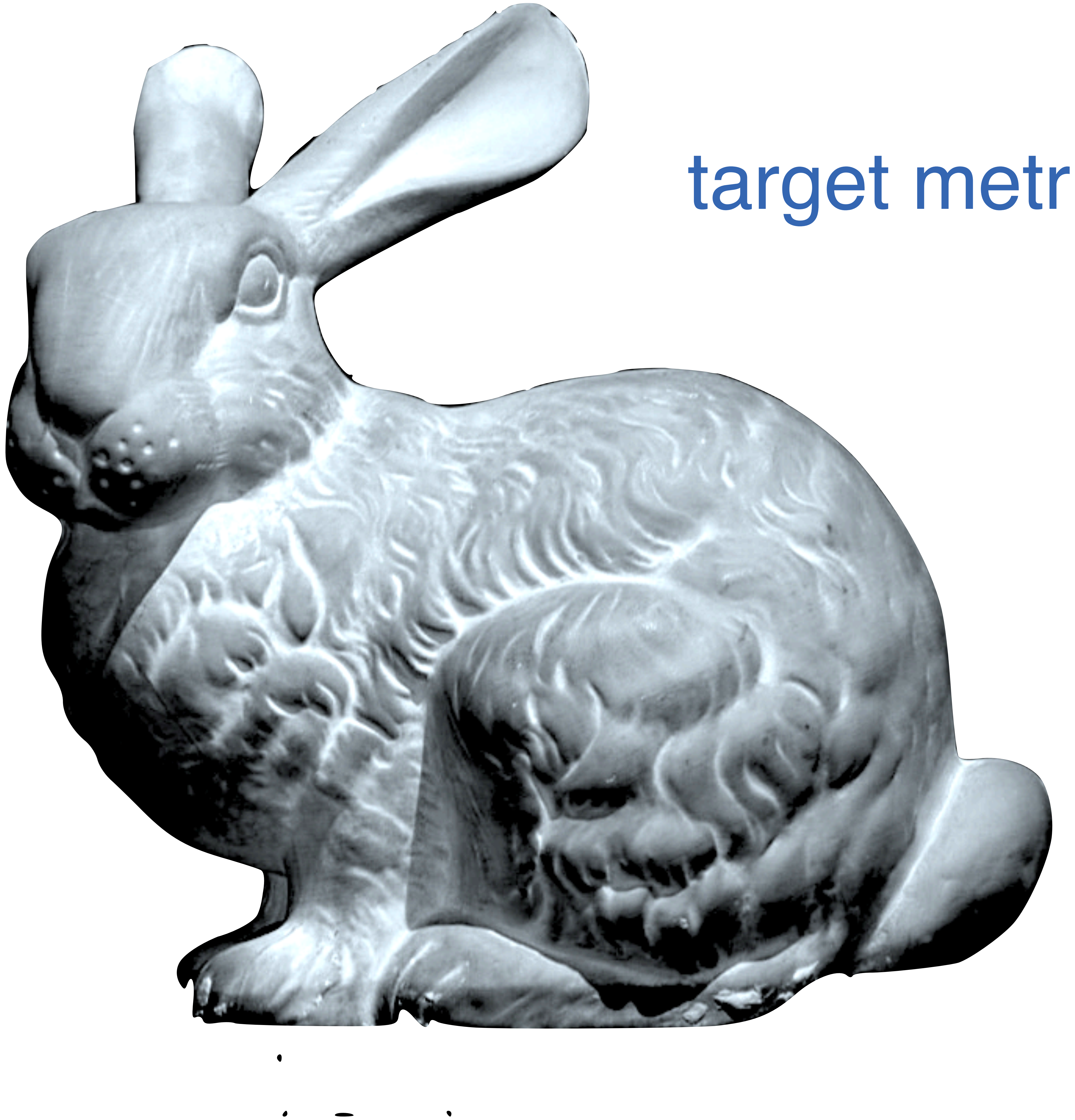
Setting up gauge field r_{ij}

Macroscopic scale

minimize $$\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_\epsilon^2$$

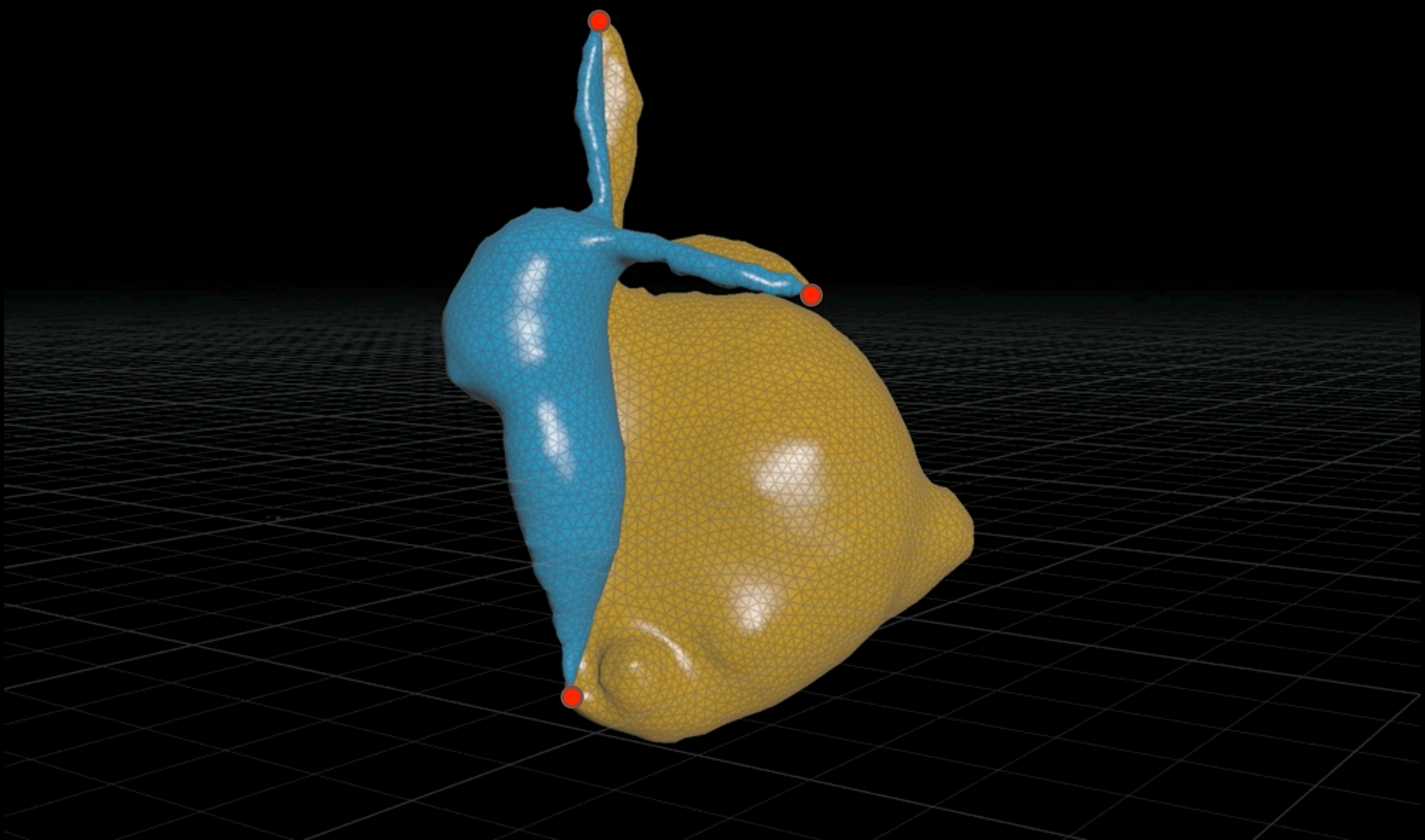


The bunny metric



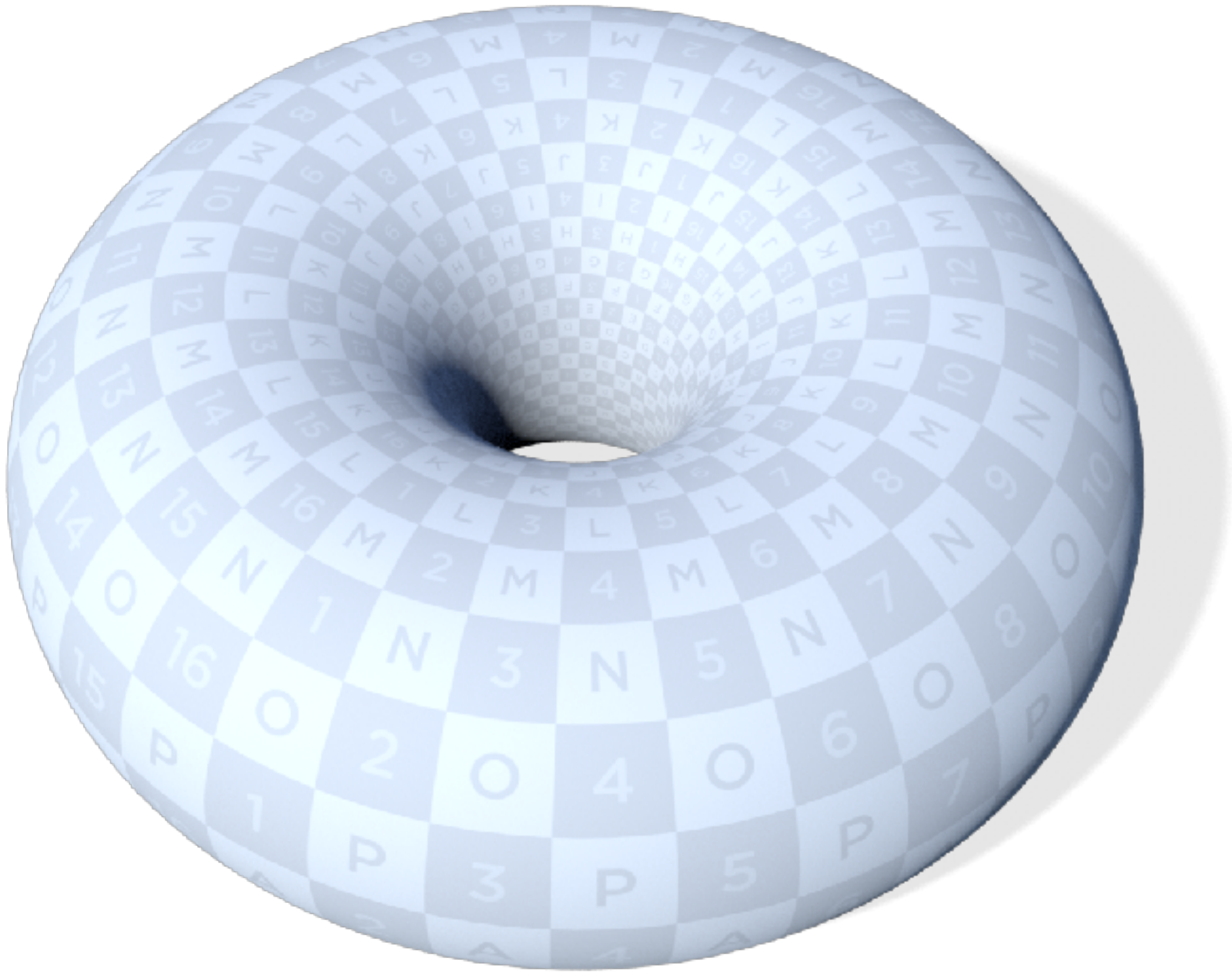
target metric

The bunny metric

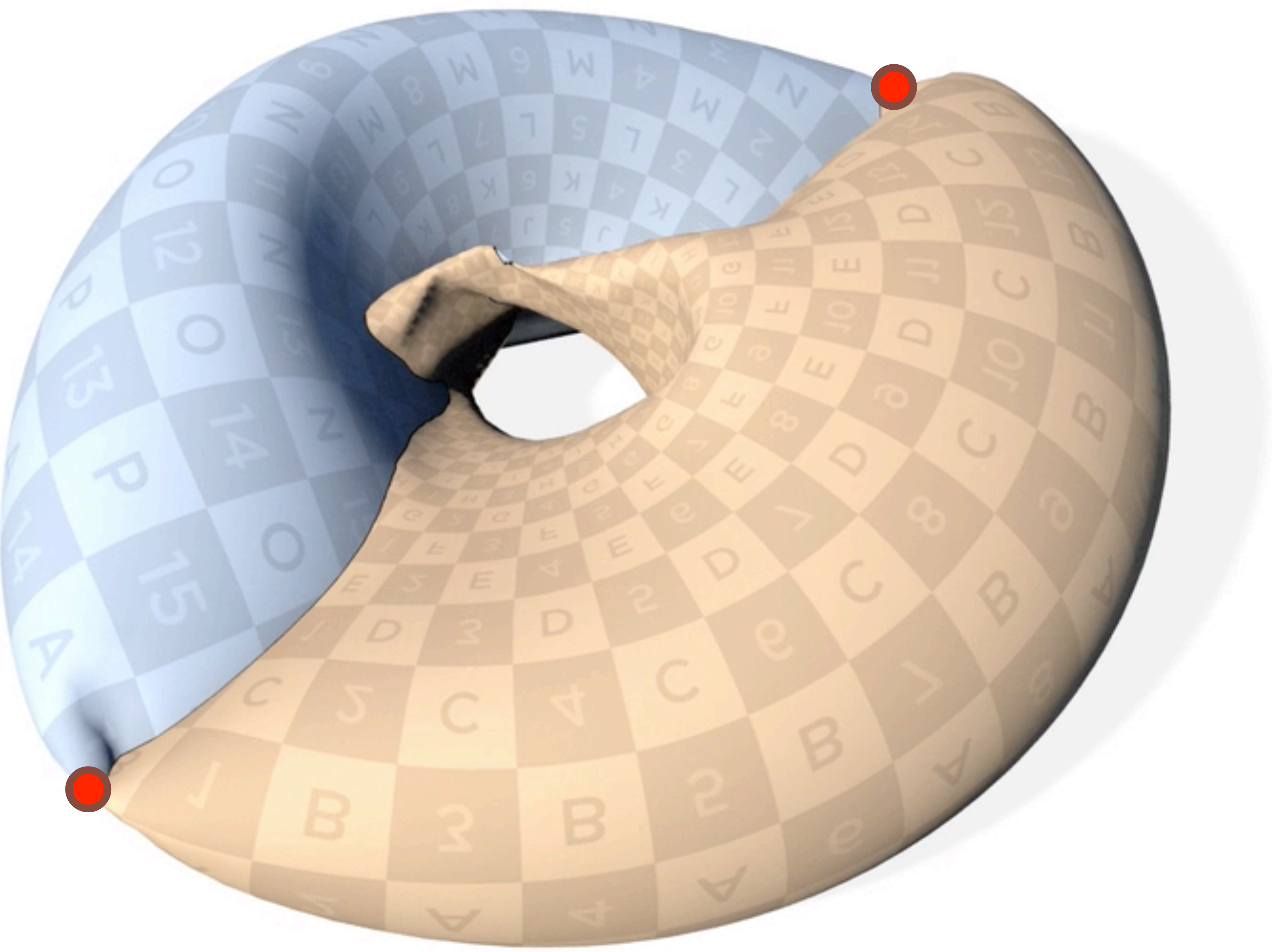


The round torus metric

target metric



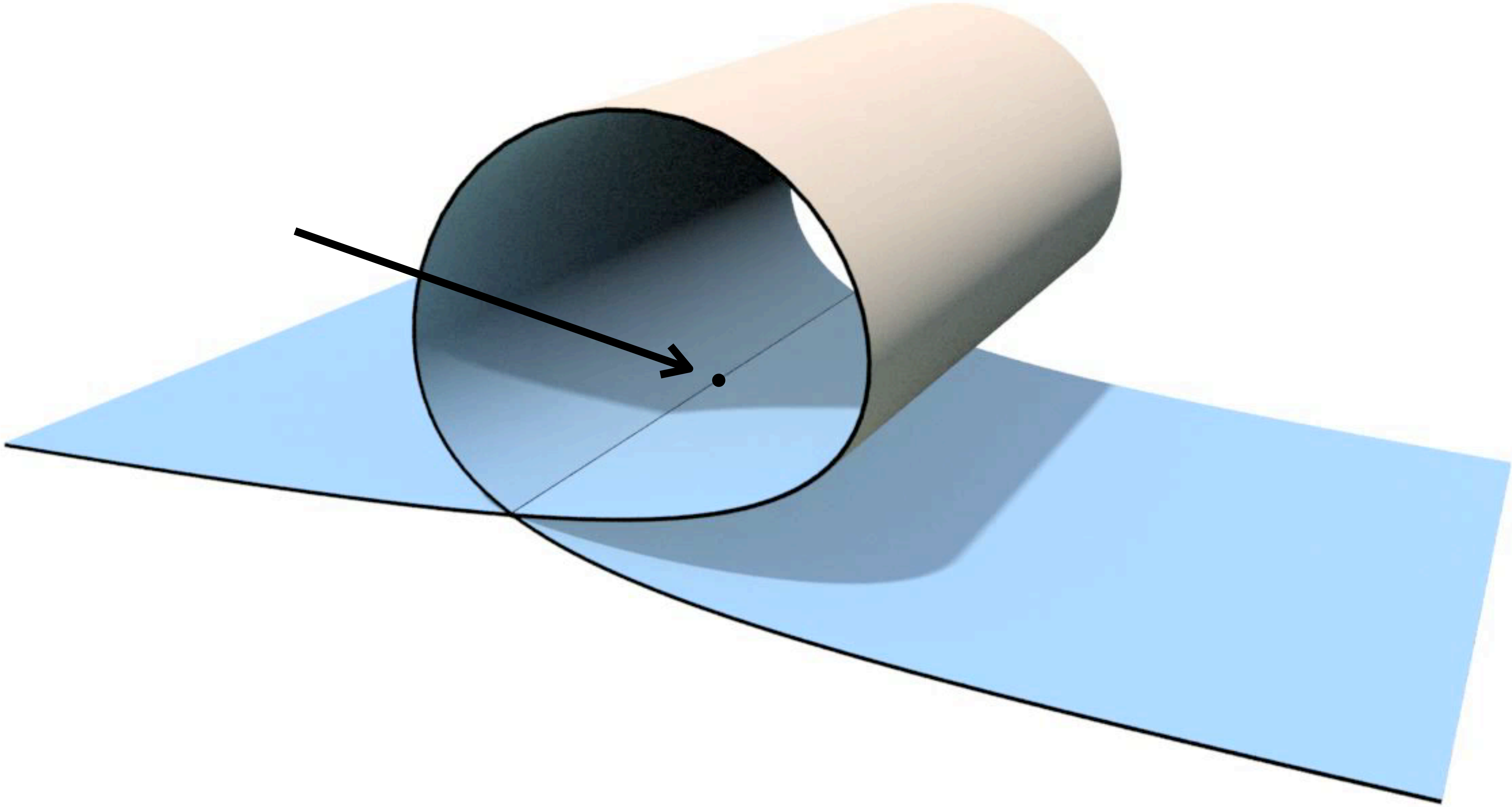
The round torus metric



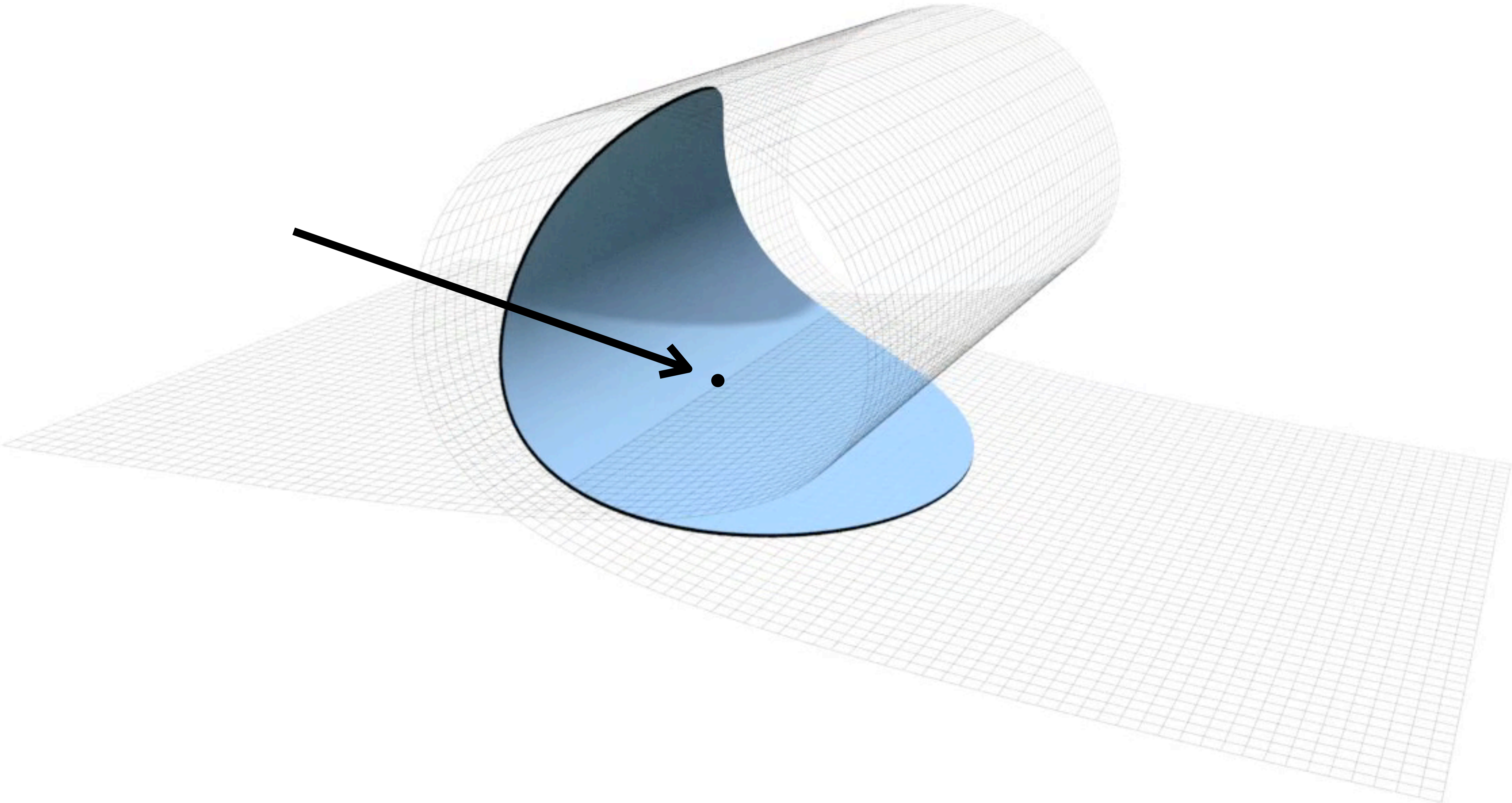
Immersion

Locally Embedded Surfaces

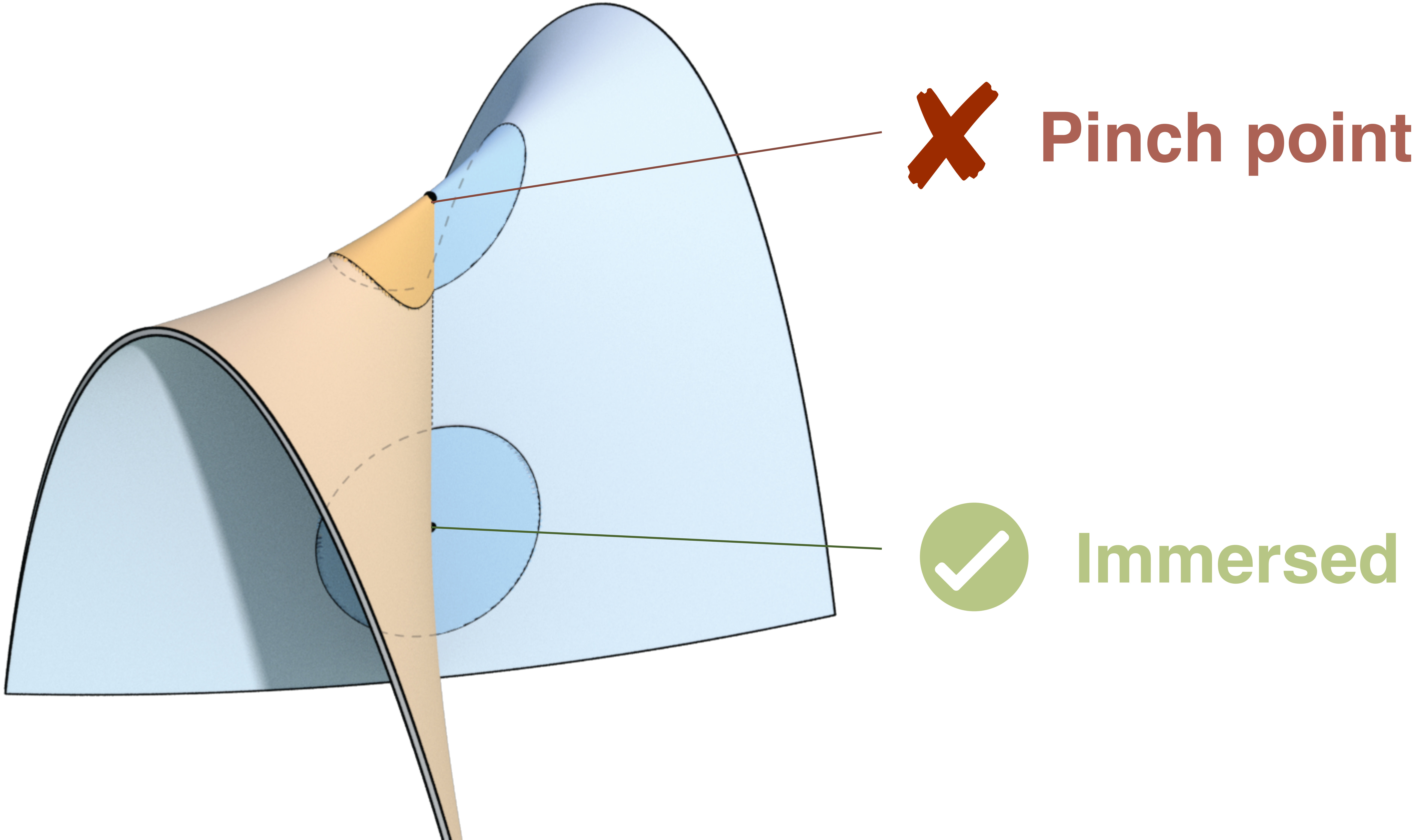
Immersion



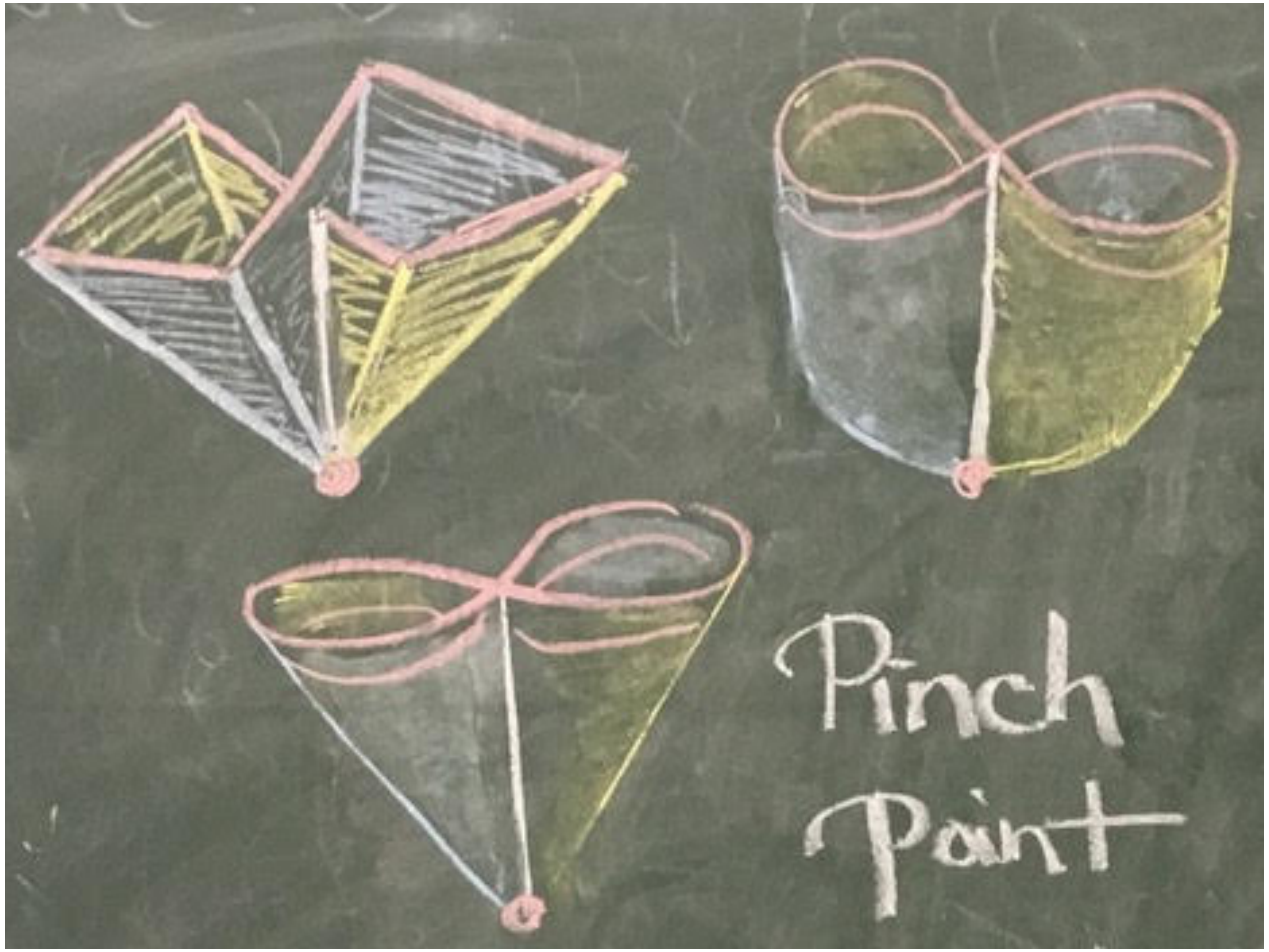
Immersion



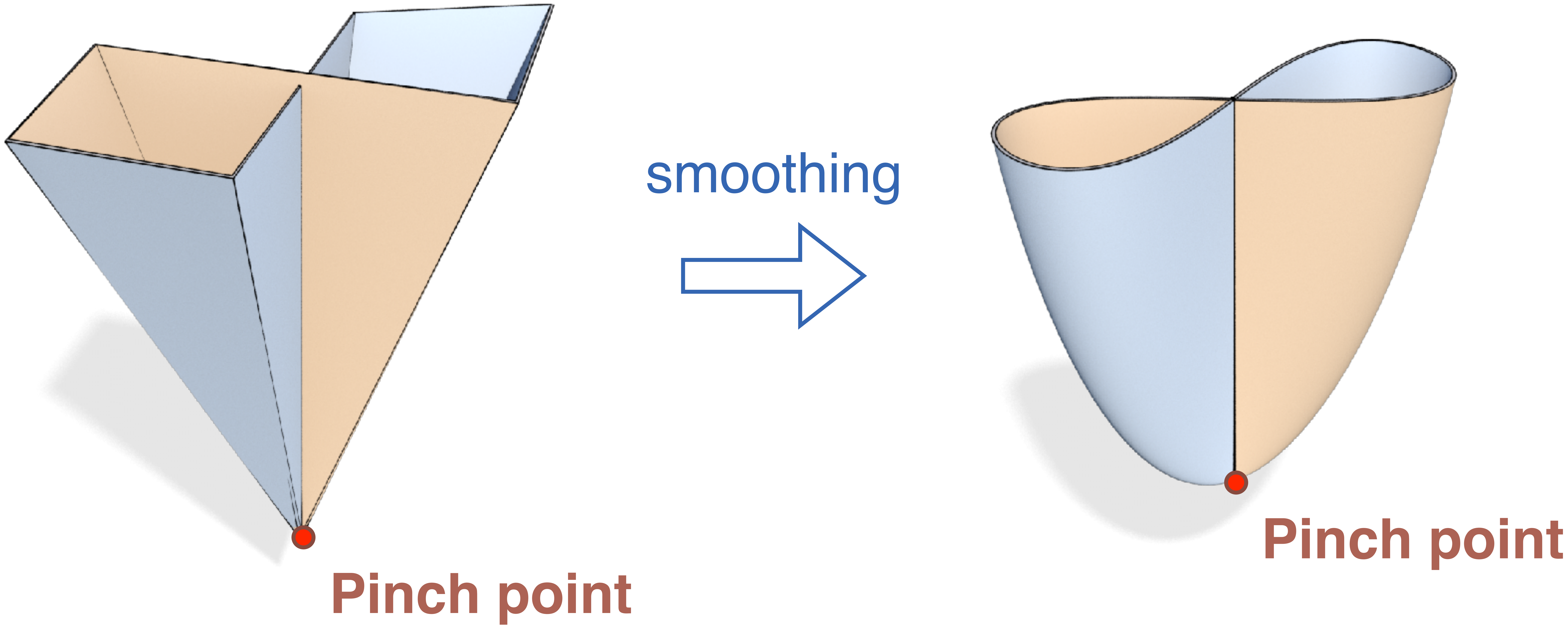
Immersion



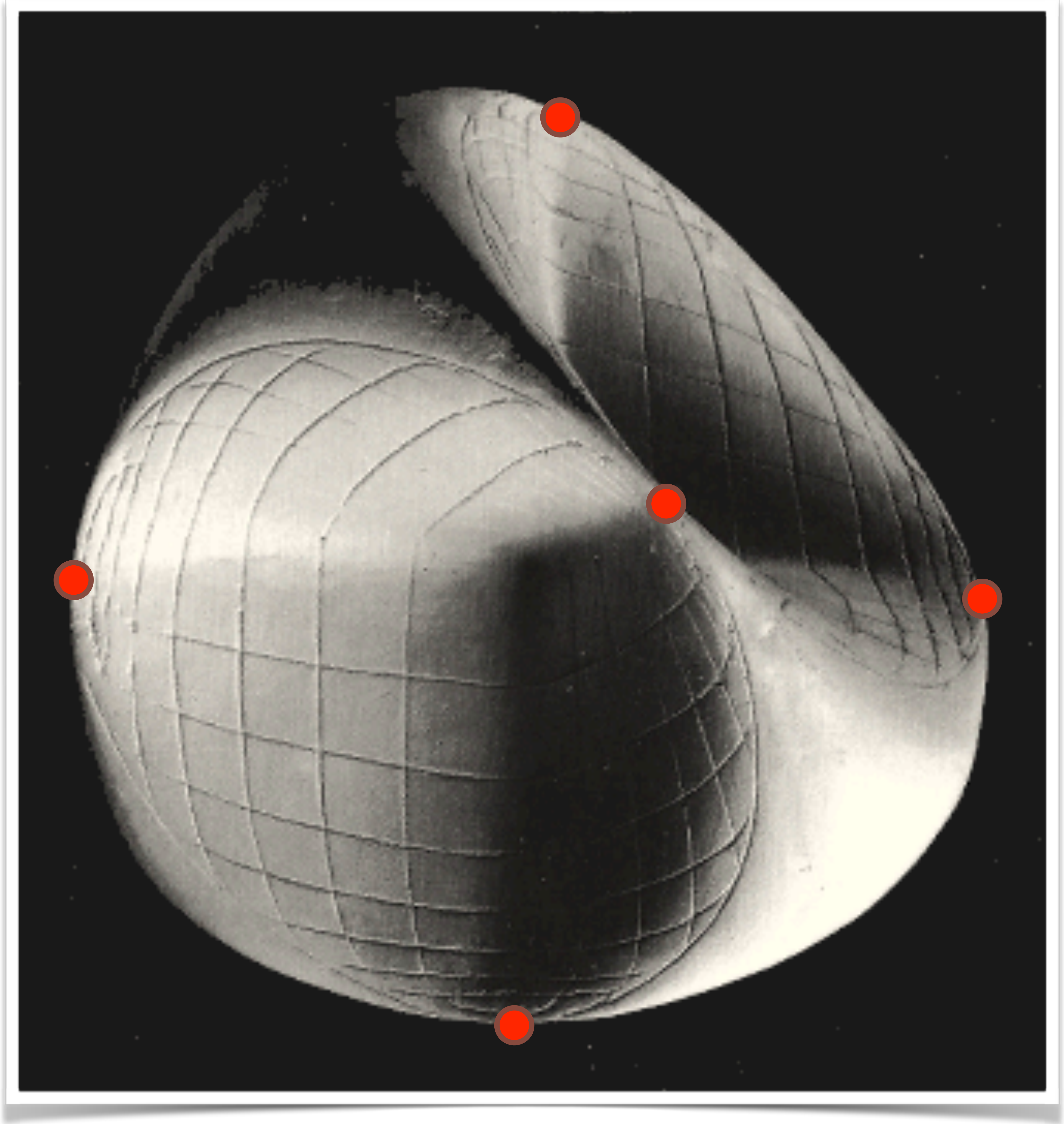
Pinch points



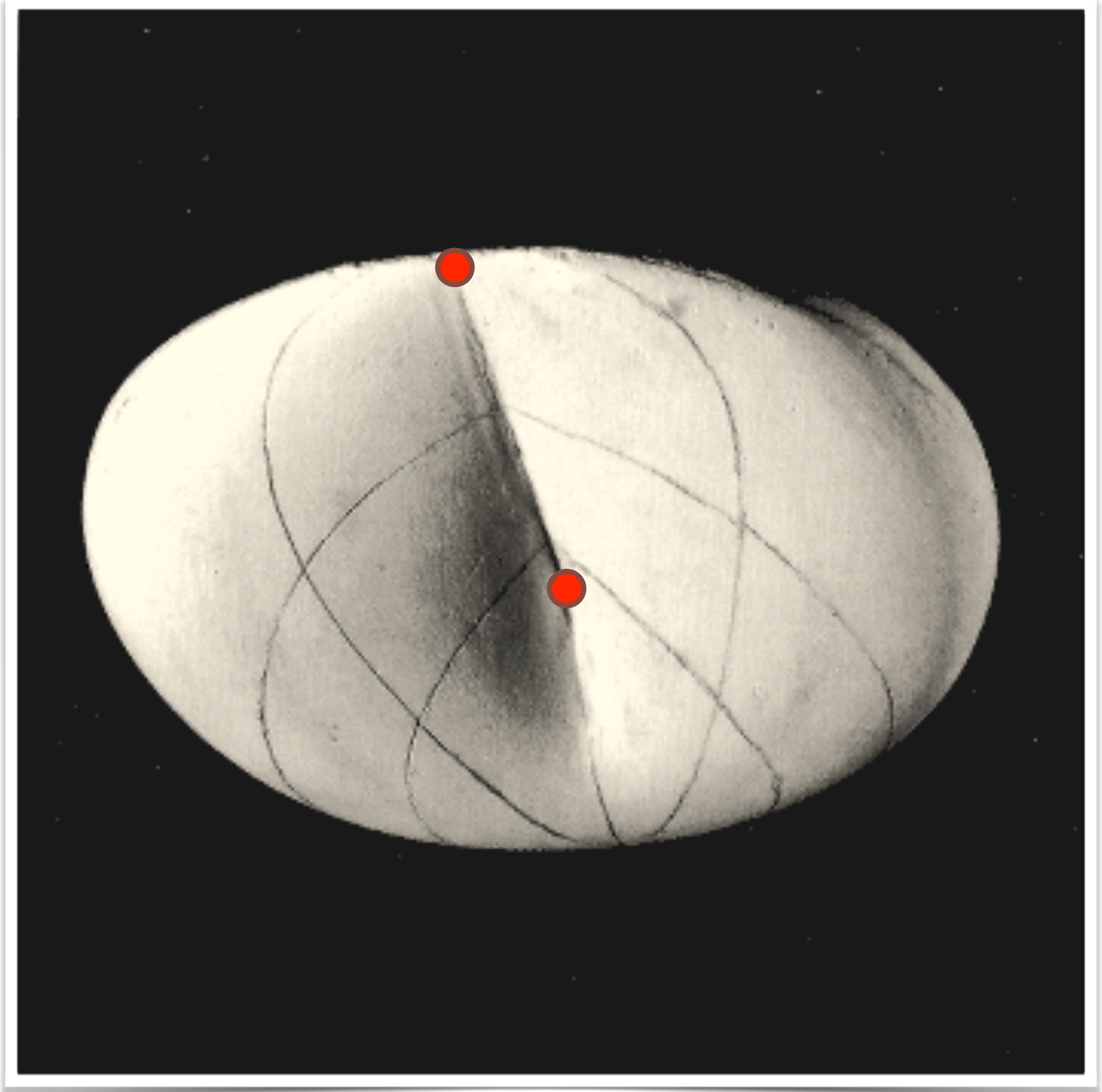
Pinch points



Pinch points

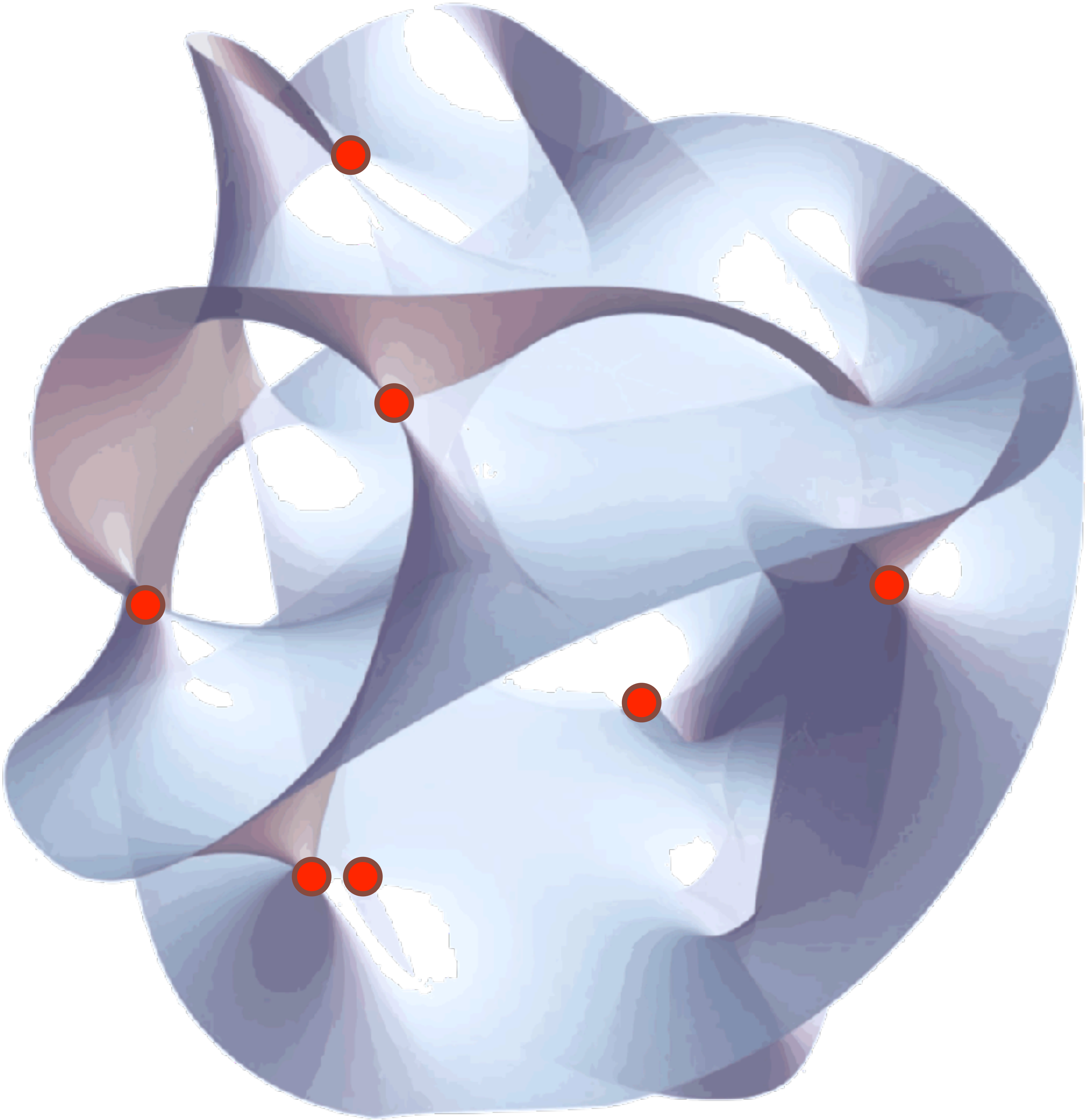


Steiner surface

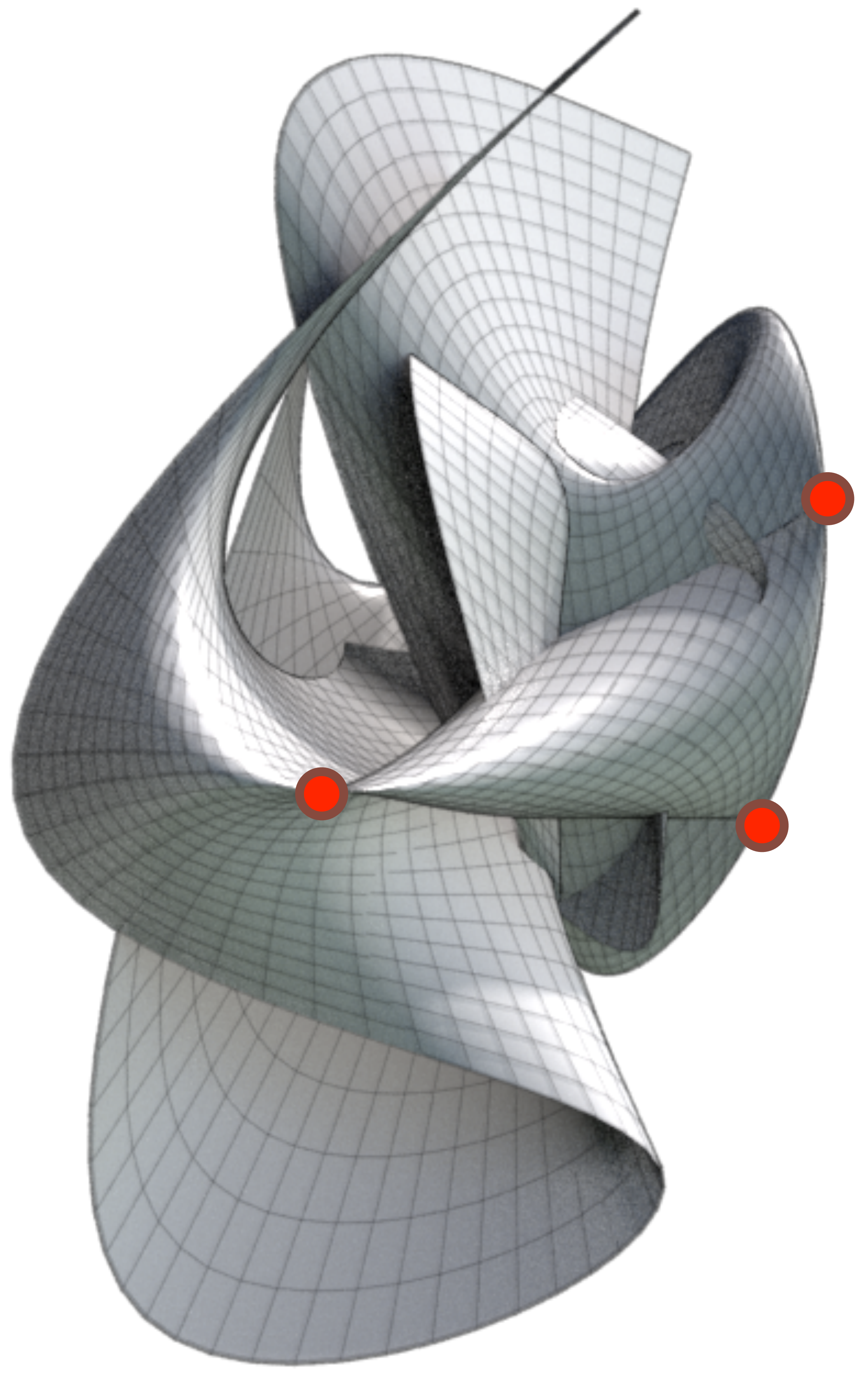


cross cap

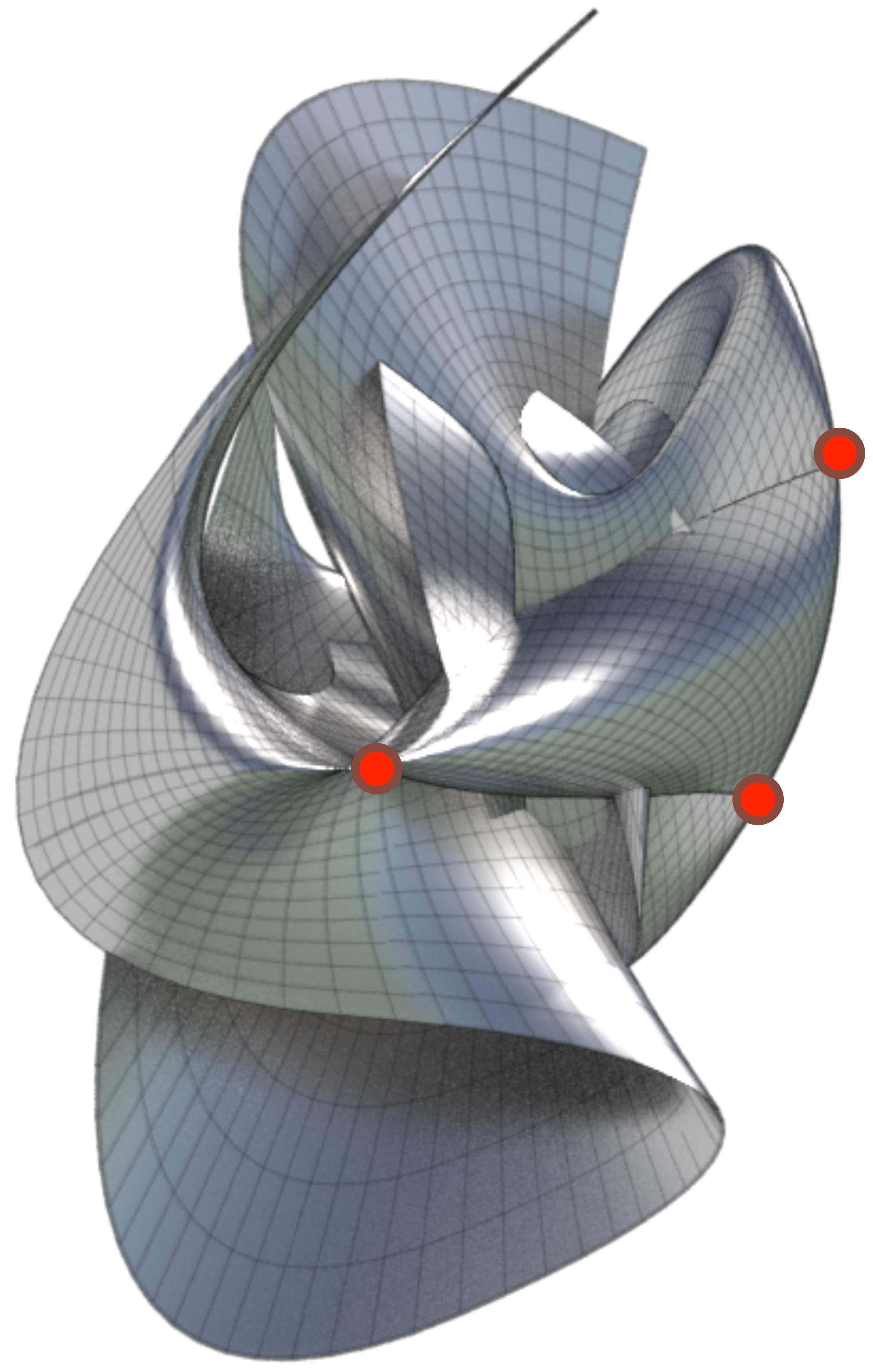
Pinch points



Pinch points



subdivision surface



NURBS surface

Pinch points

Emergent surface

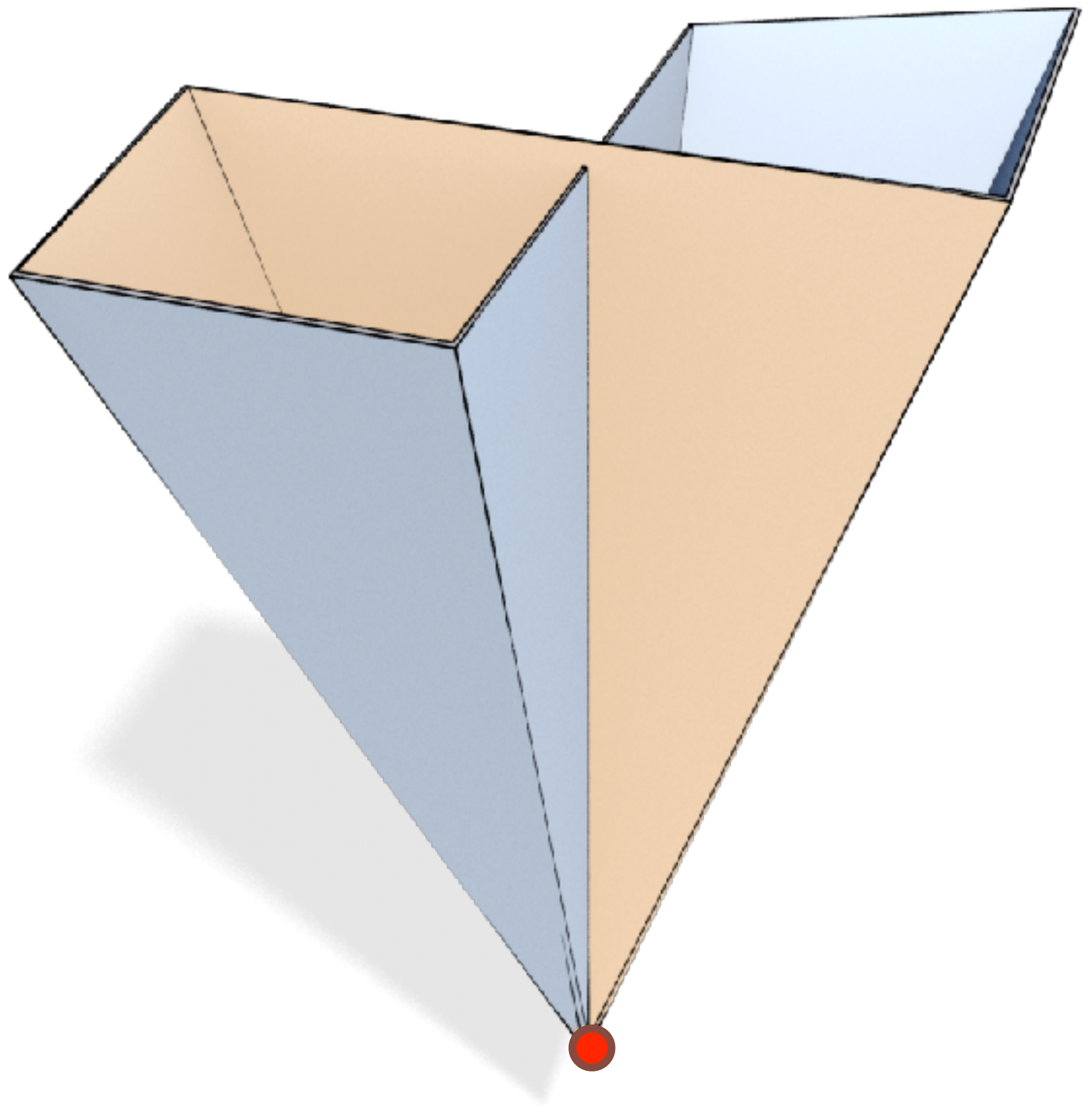
Microscopic scale

Setting up gauge field r_{ij}

Macroscopic scale

minimize $\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_\epsilon^2$

Invisible to pinch points



Emergent surface

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Setting up gauge field r_{ij}

Macroscopic scale

minimize $\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_{\epsilon}^{2}$

Invisible to pinch points

Can we ensure ***immersion*** *for such emergent isometric surfaces?*

Emergent surface

Microscopic scale

Setting up gauge field r_{ij}

Macroscopic scale

minimize $\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_\epsilon^2$

Invisible to pinch points

Can we ensure ***immersion*** *for such emergent isometric surfaces?*

YES

Descriptions of rotations

Rotation matrices $\mathrm{SO}(3)$

$$Q \in \mathbb{R}^{3\times 3}, \quad Q^\intercal Q = I, \quad \det(Q) = 1$$

3D rotation $\mathbf{v} \mapsto Q\mathbf{v}$

Unit quaternions $\mathrm{SU}(2)$

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}, \quad |q| = 1$$

3D rotation $\mathbf{v} \mapsto q\mathbf{v}\bar{q}$

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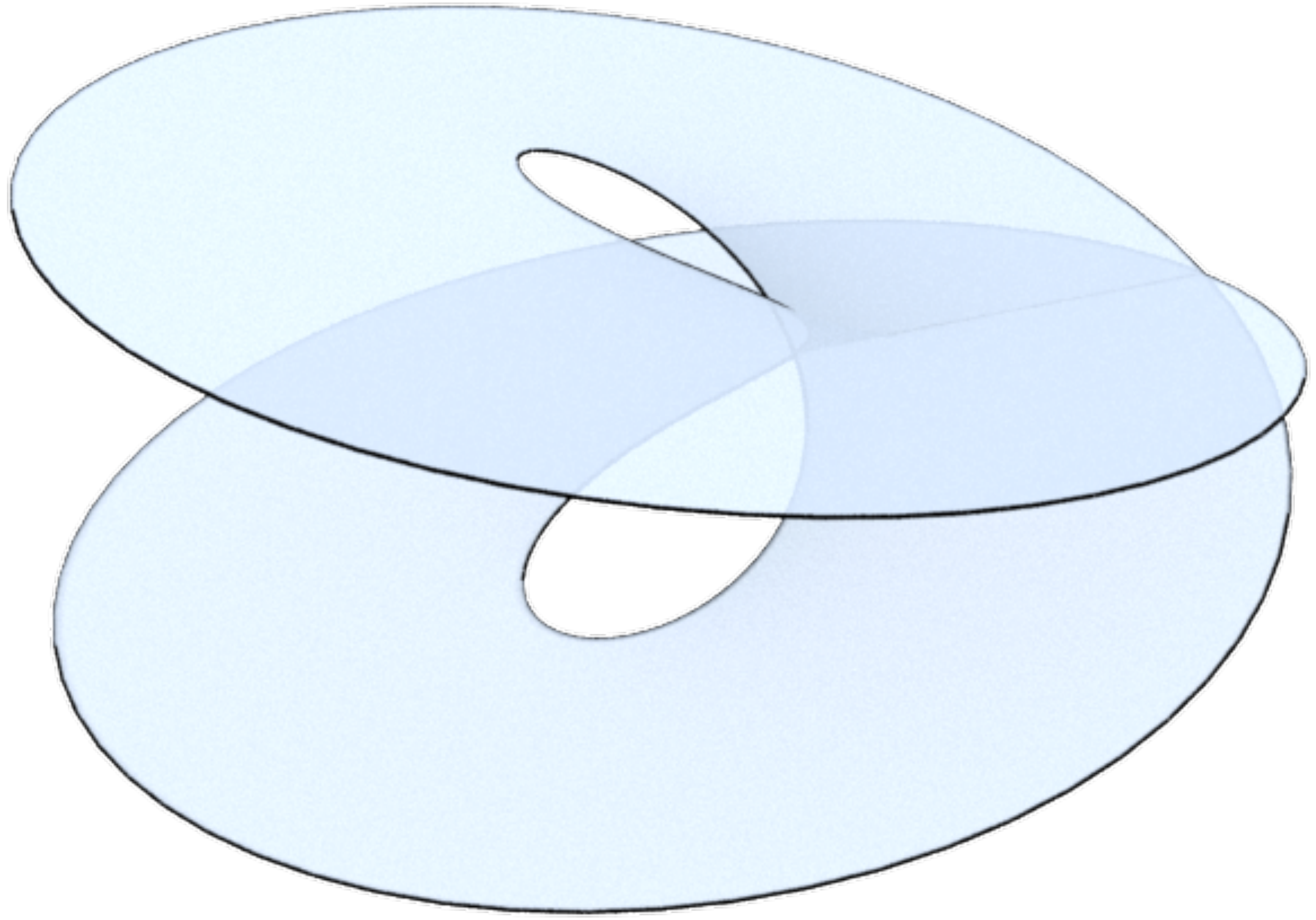
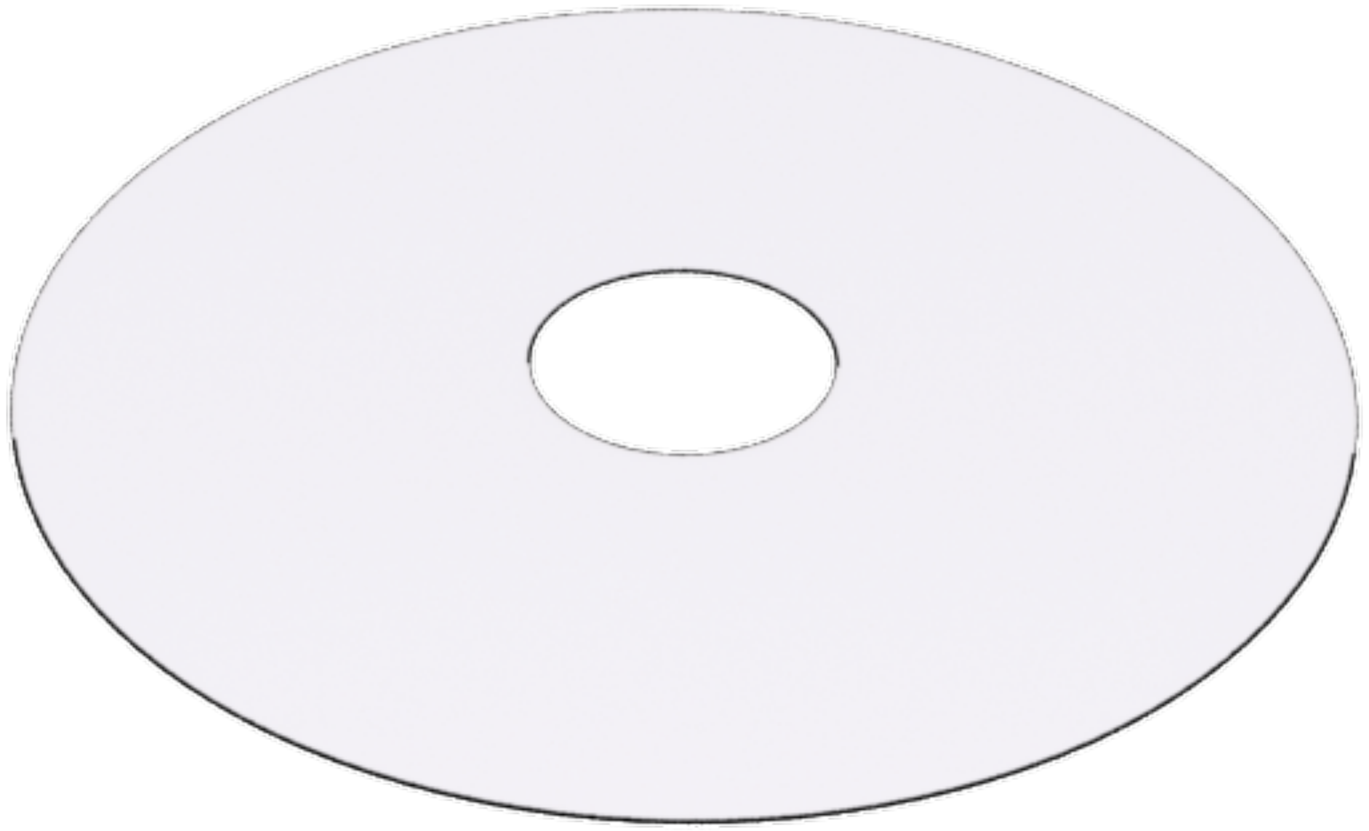
3D rotation $\mathbf{v} \mapsto \boxed{q}\mathbf{v}\boxed{\bar{q}}$ *square root of the rotation*

$q, -q$ *represent the same rotation*

Descriptions of rotations

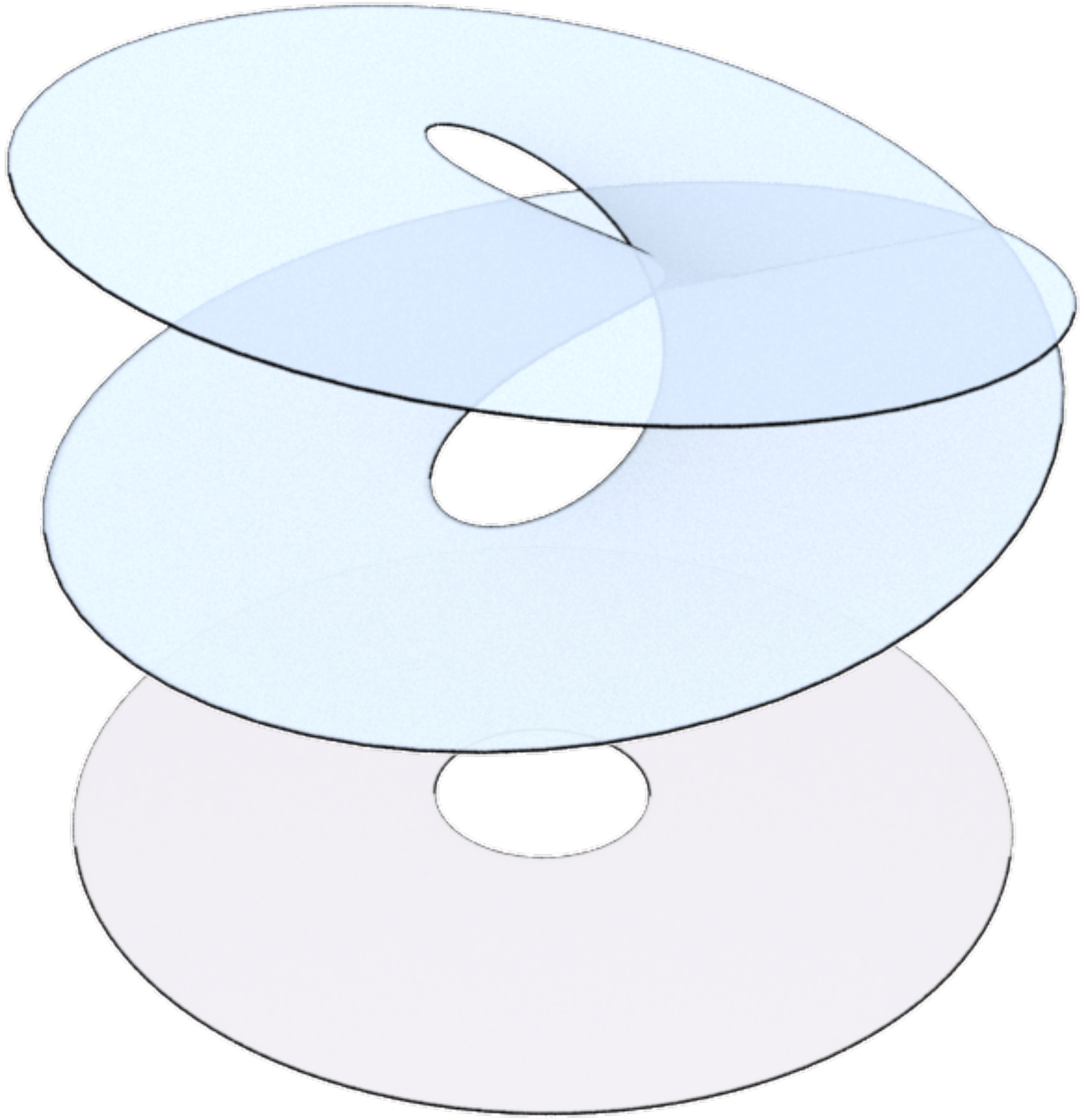


$\mathrm{SU}(2)$
unit quaternions



$\mathrm{SO}(3)$
rotation matrices

Descriptions of rotations



$\mathrm{SU}(2)$
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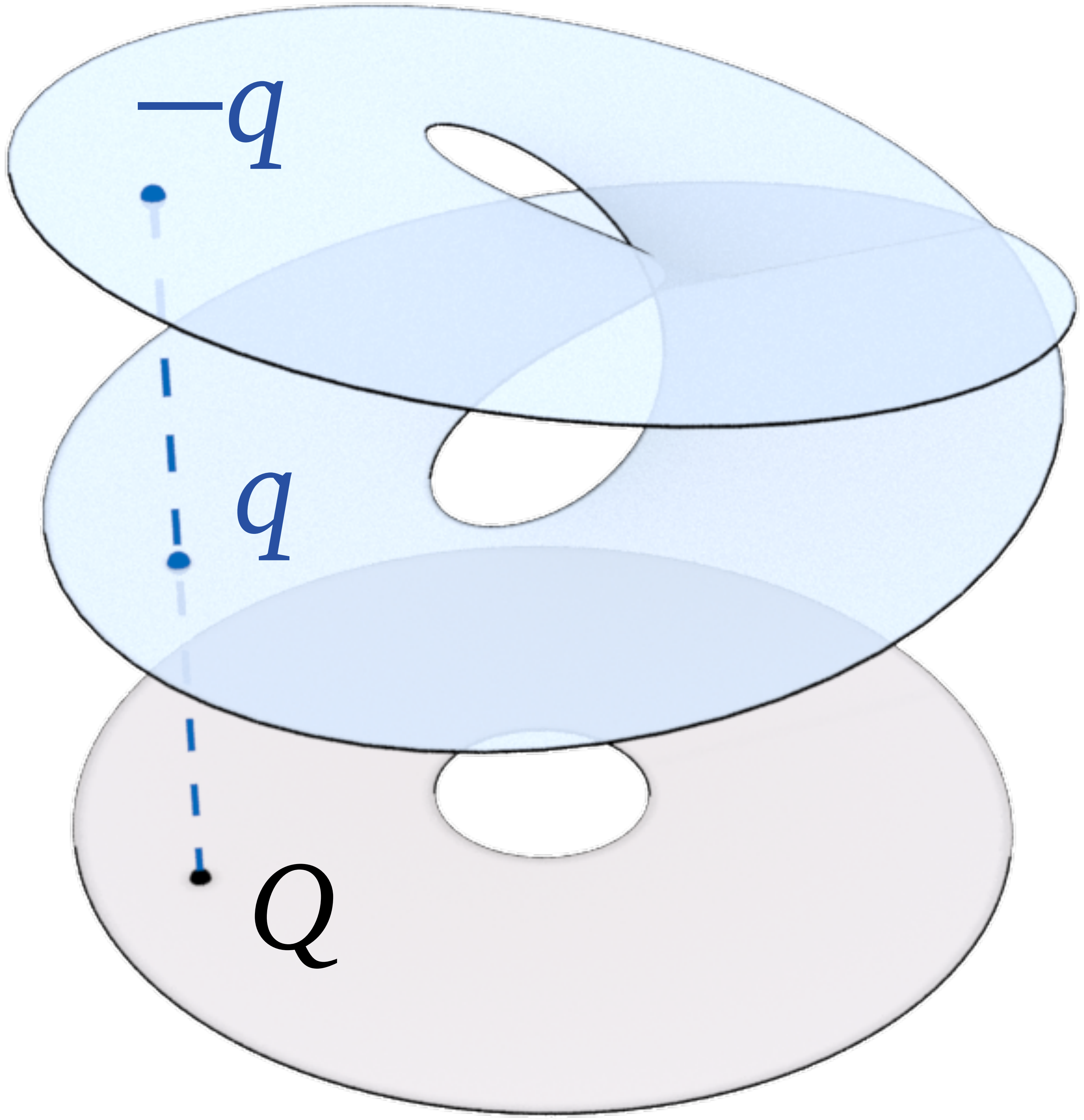

$\mathrm{SU}(2)$
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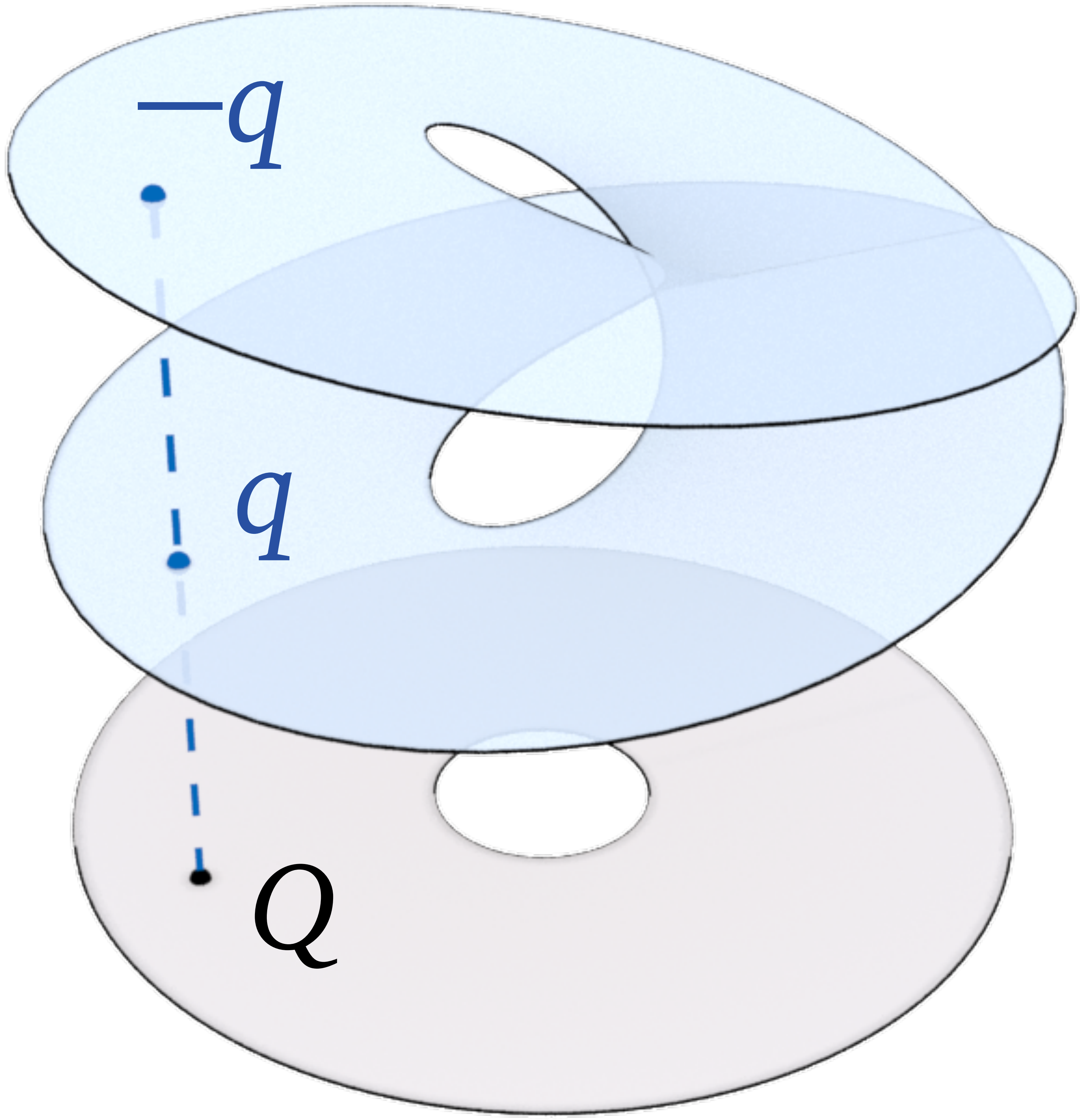


SU(2)
unit quaternions
"spinors"

SO(3)
rotation matrices
"rotations"

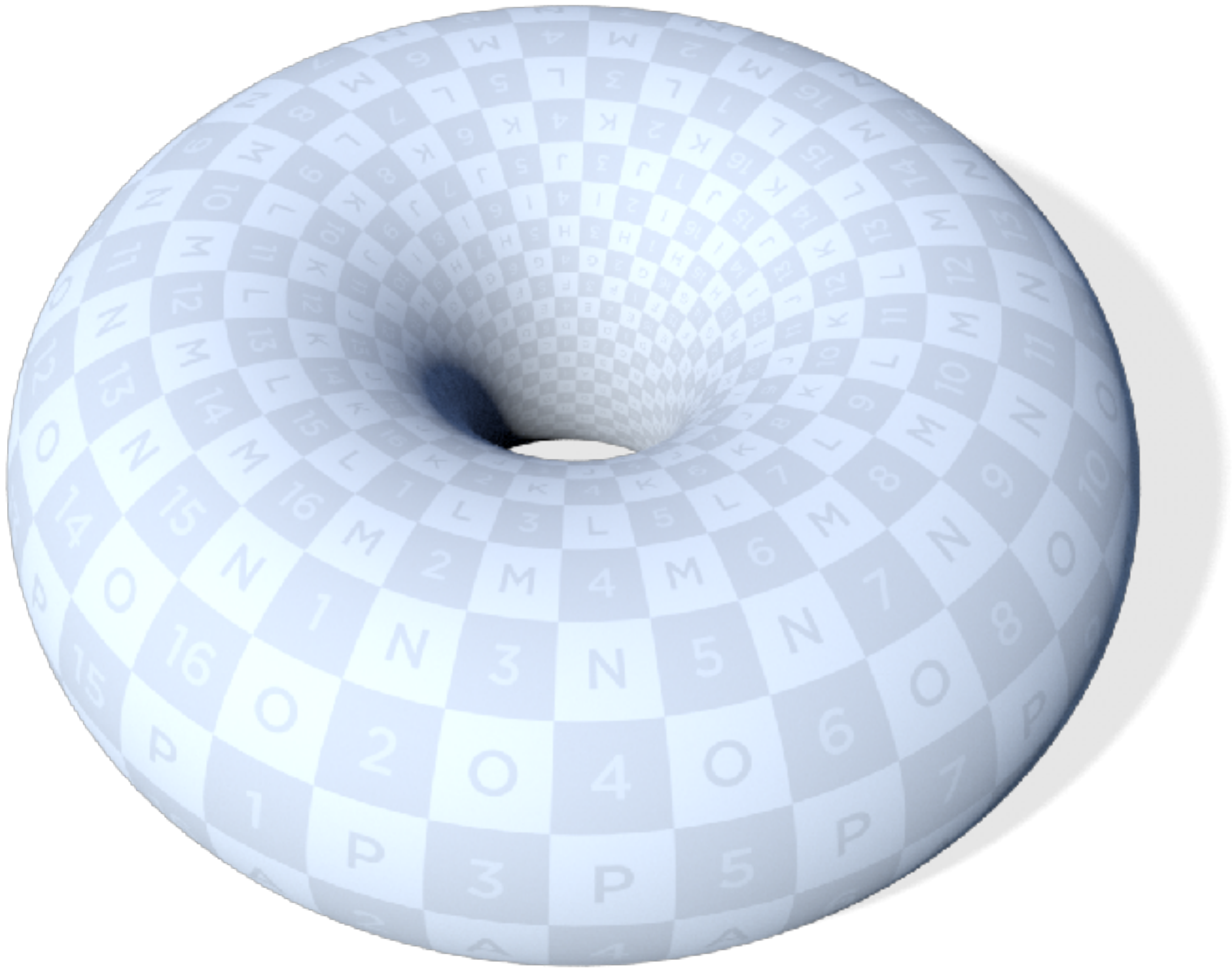
Descriptions of rotations

rotation matrices
"rotations"

unit quaternions
"spinors"

Descriptions of rotations

target metric



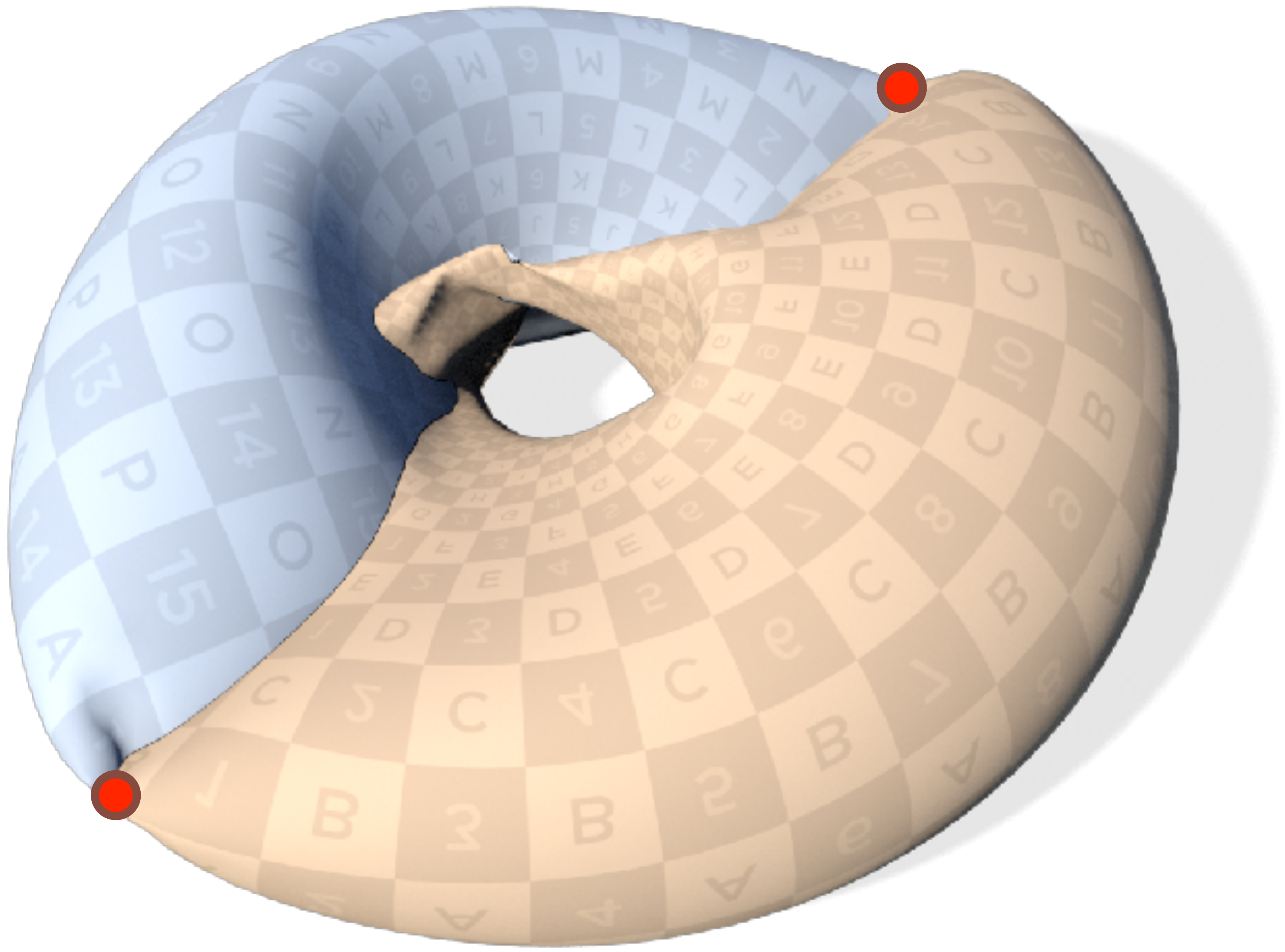
rotation matrices

"rotations"

unit quaternions

"spinors"

Descriptions of rotations



rotation matrices

"rotations"

unit quaternions

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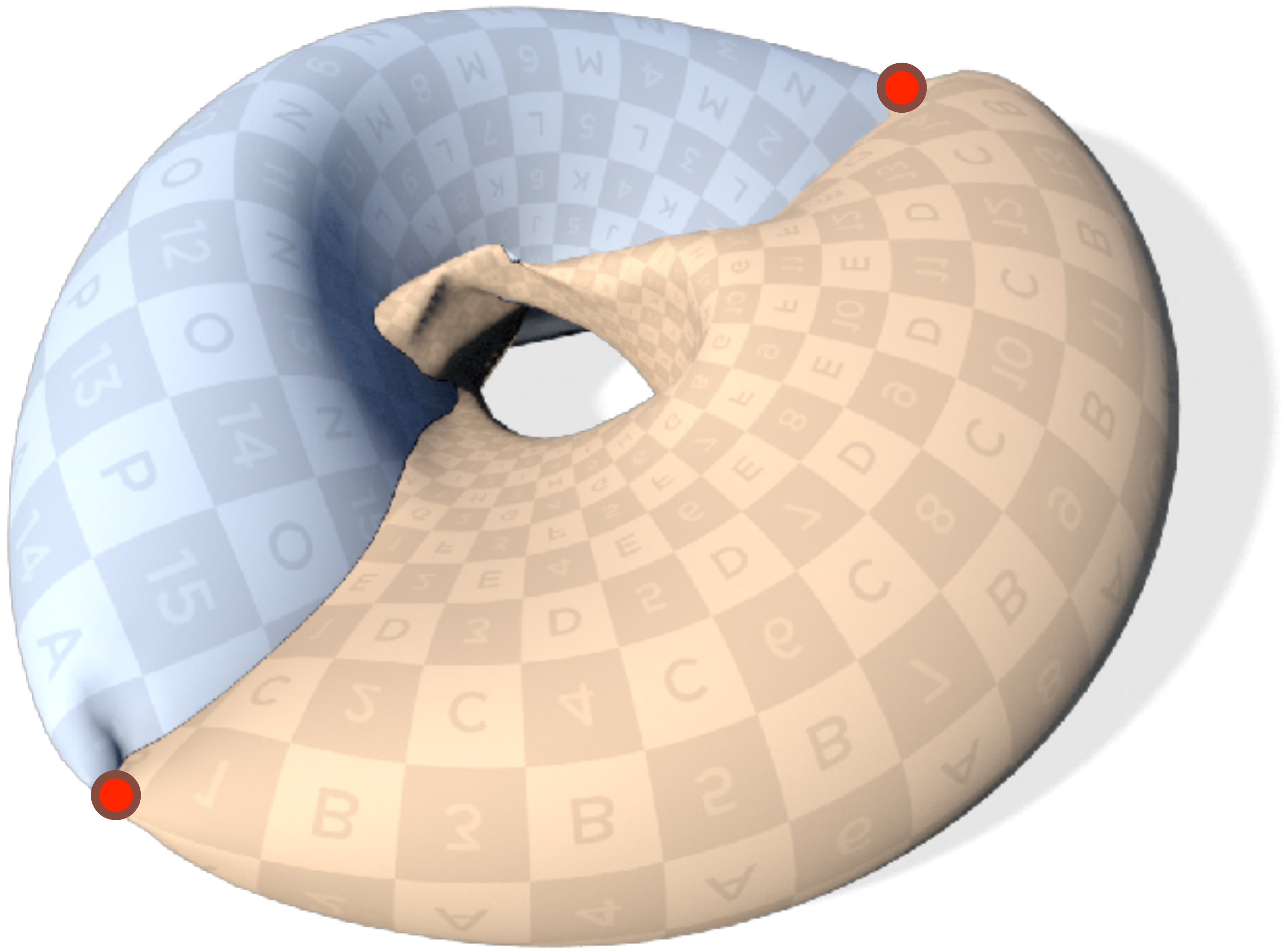
Descriptions of rotations



rotation matrices
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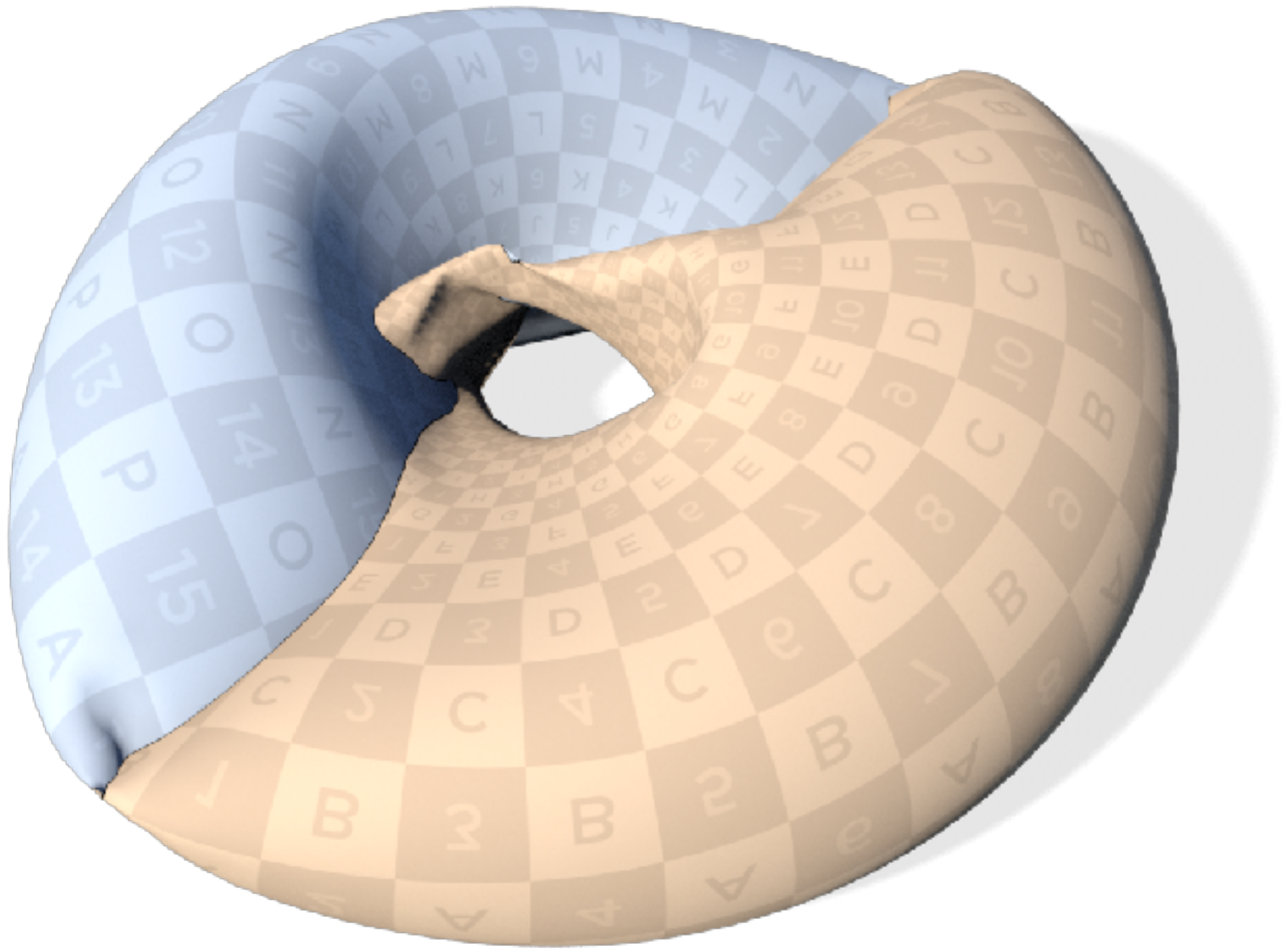
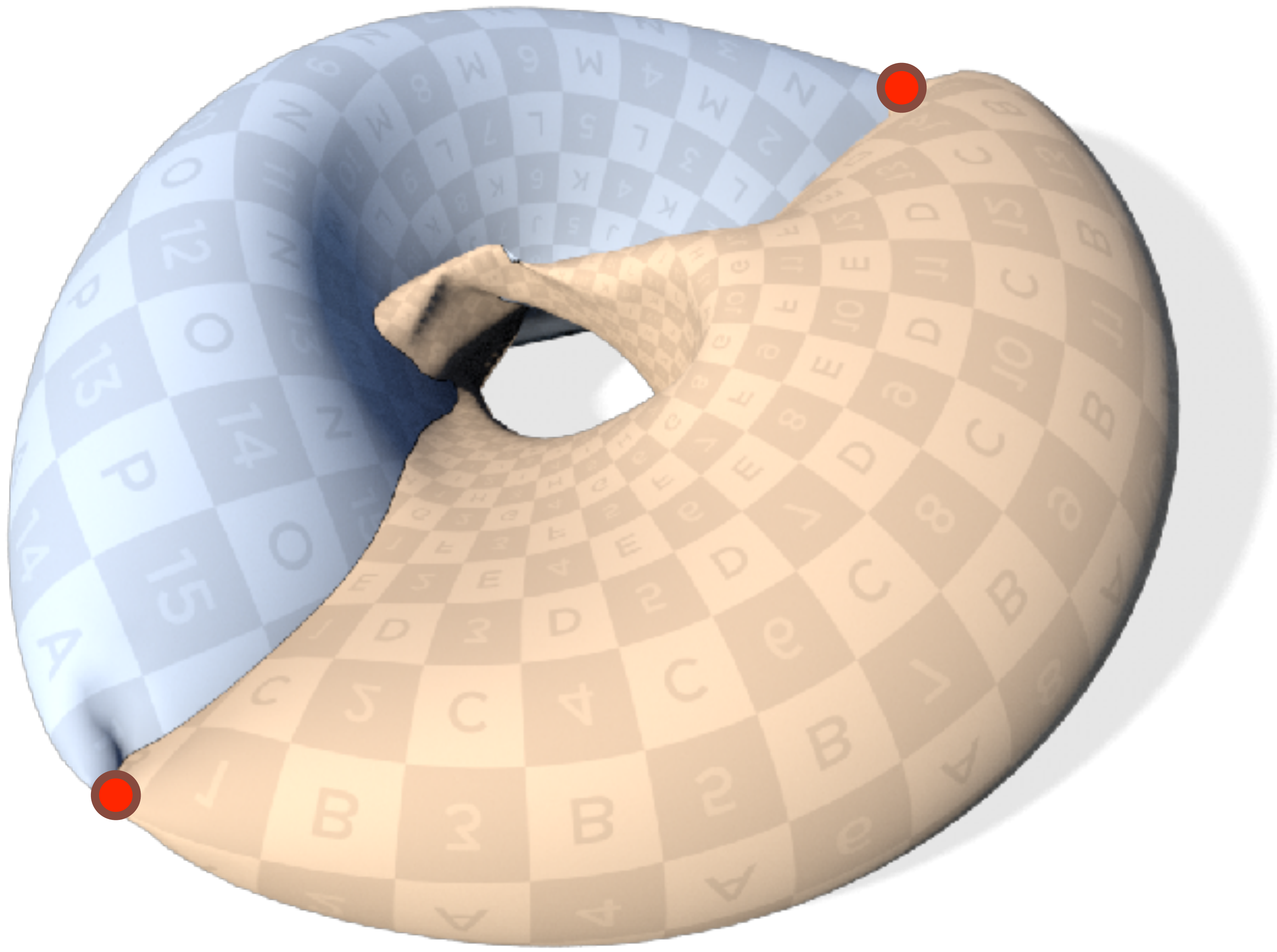
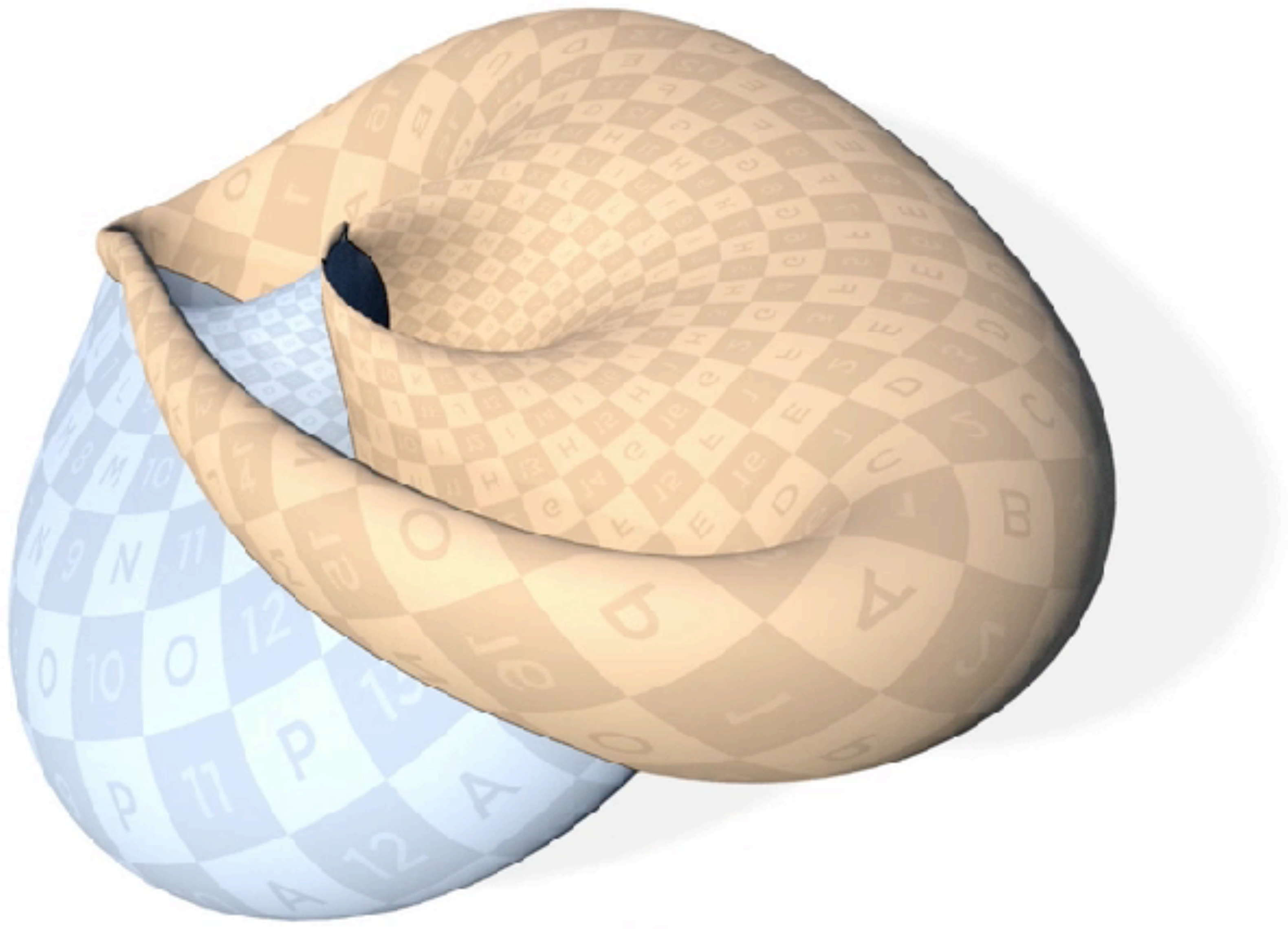


unit quaternions
"spinors"

Descriptions of rotations



rotation matrices
"rotations"



unit quaternions
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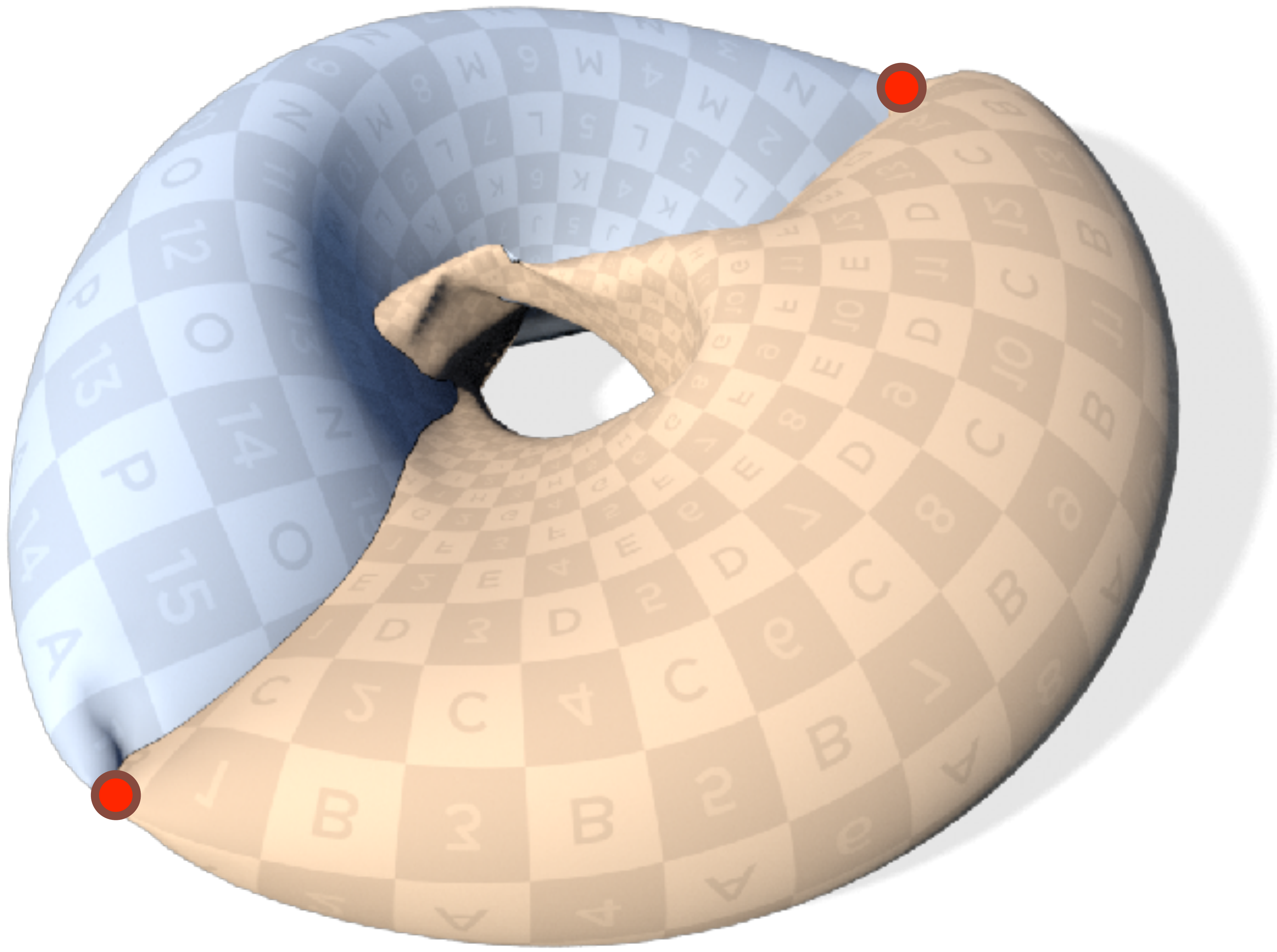
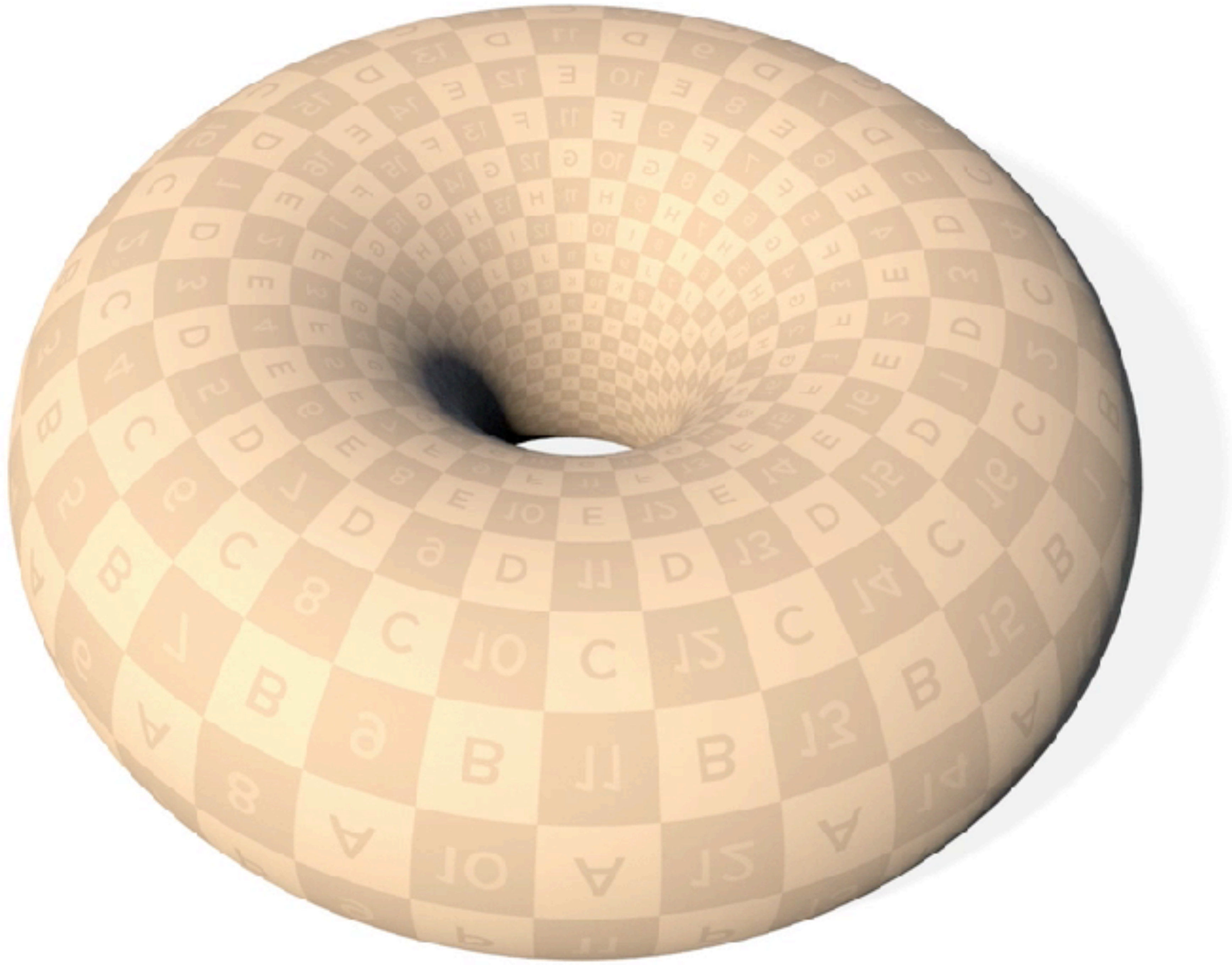
Descriptions of rotations

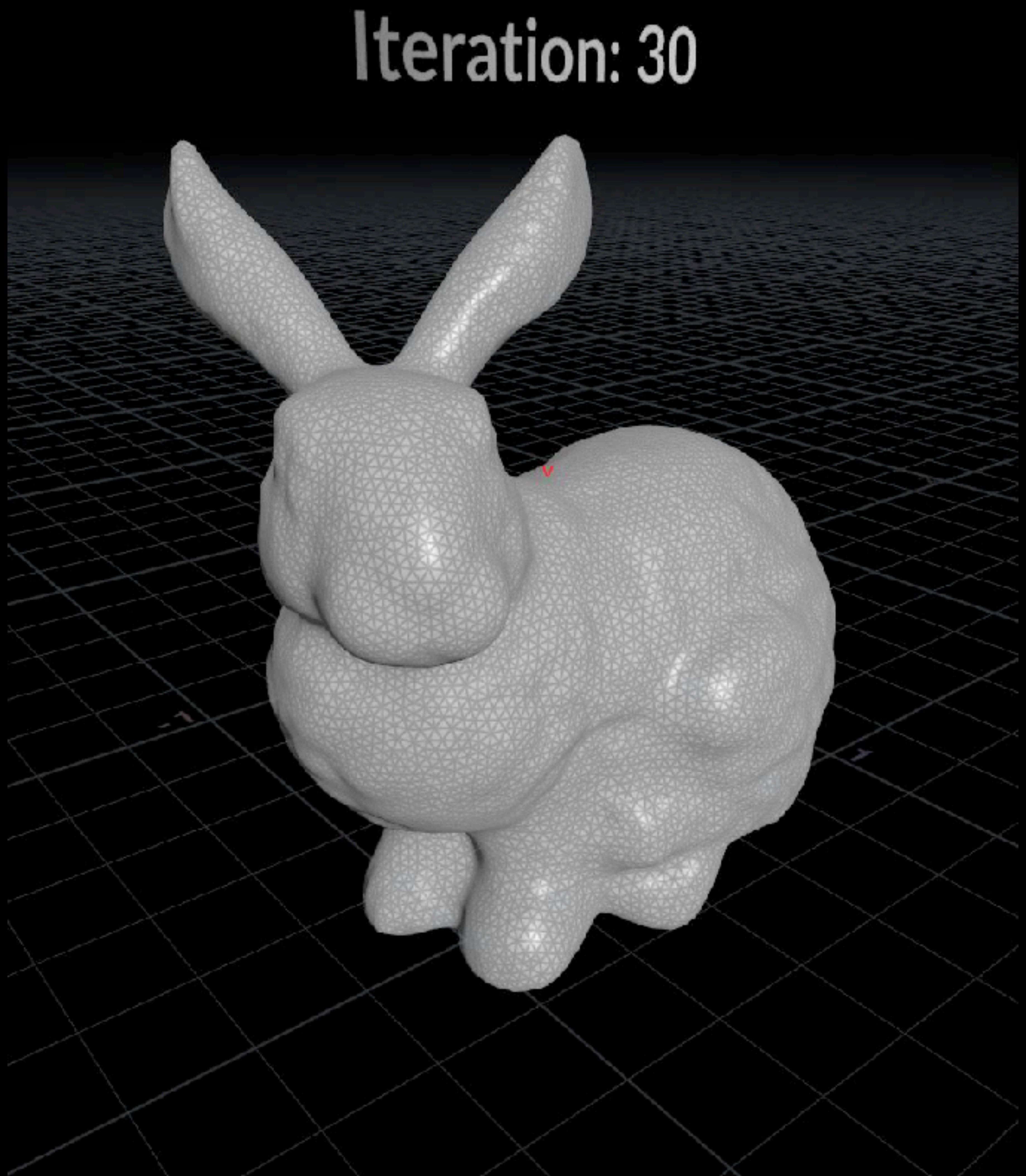


rotation matrices
"rotations"

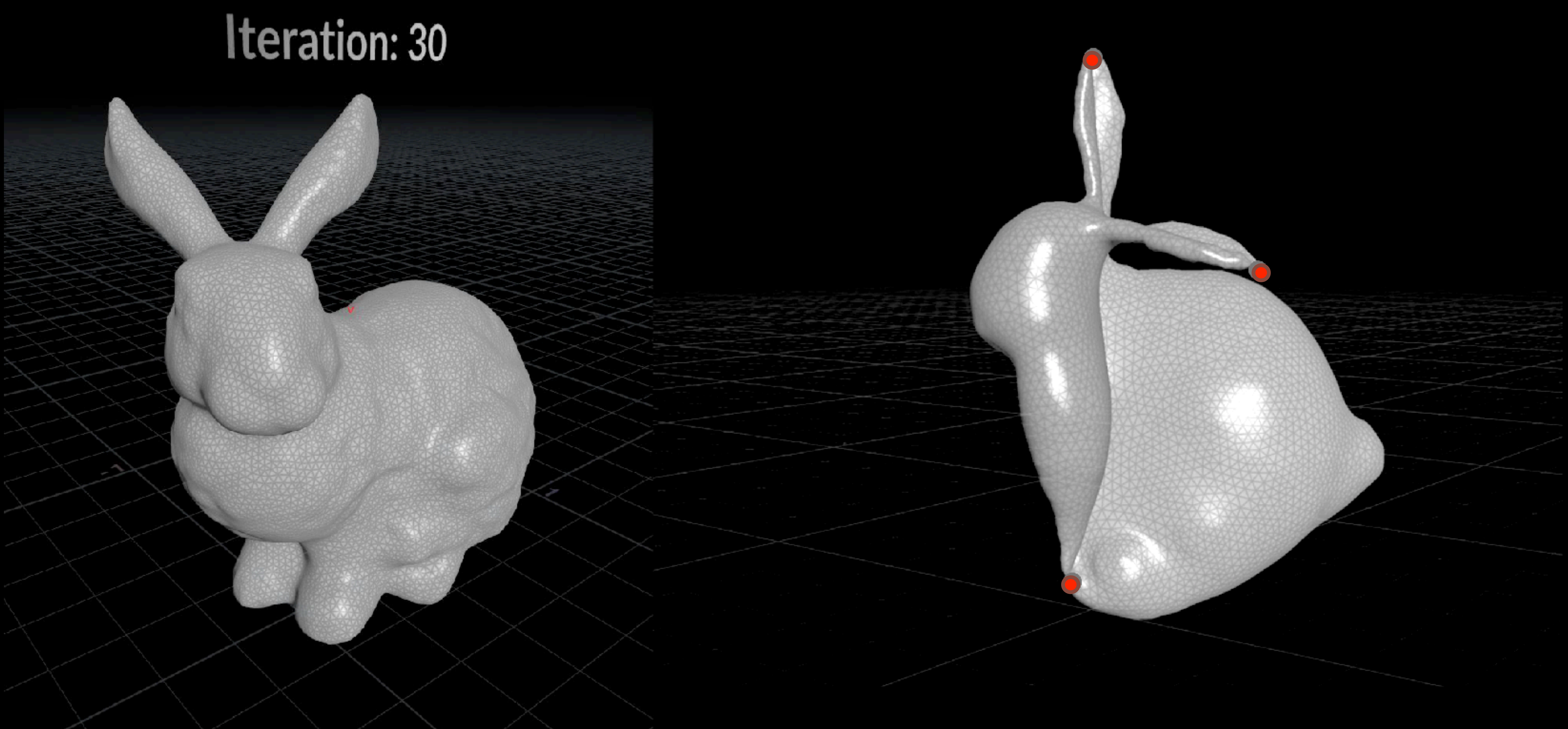


unit quaternions
"spinors"

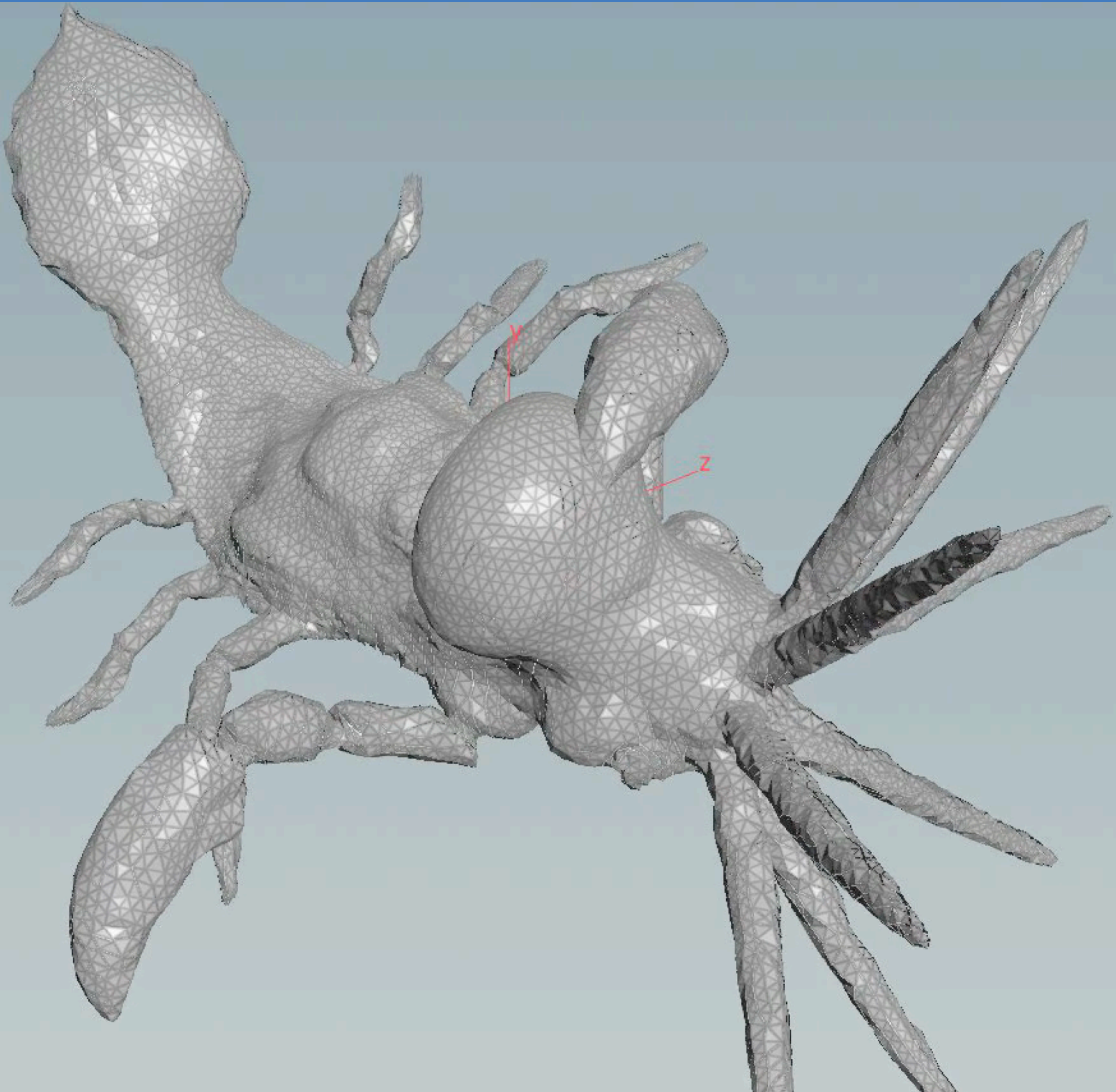
Spinorial gauge theory



Spinorial gauge theory



Spinorial gauge theory



Emergent surface

Can we ensure ***immersion*** *for such emergent isometric surfaces?*

YES

Emergent surface

Can we ensure ***immersion***
for such emergent isometric surfaces?

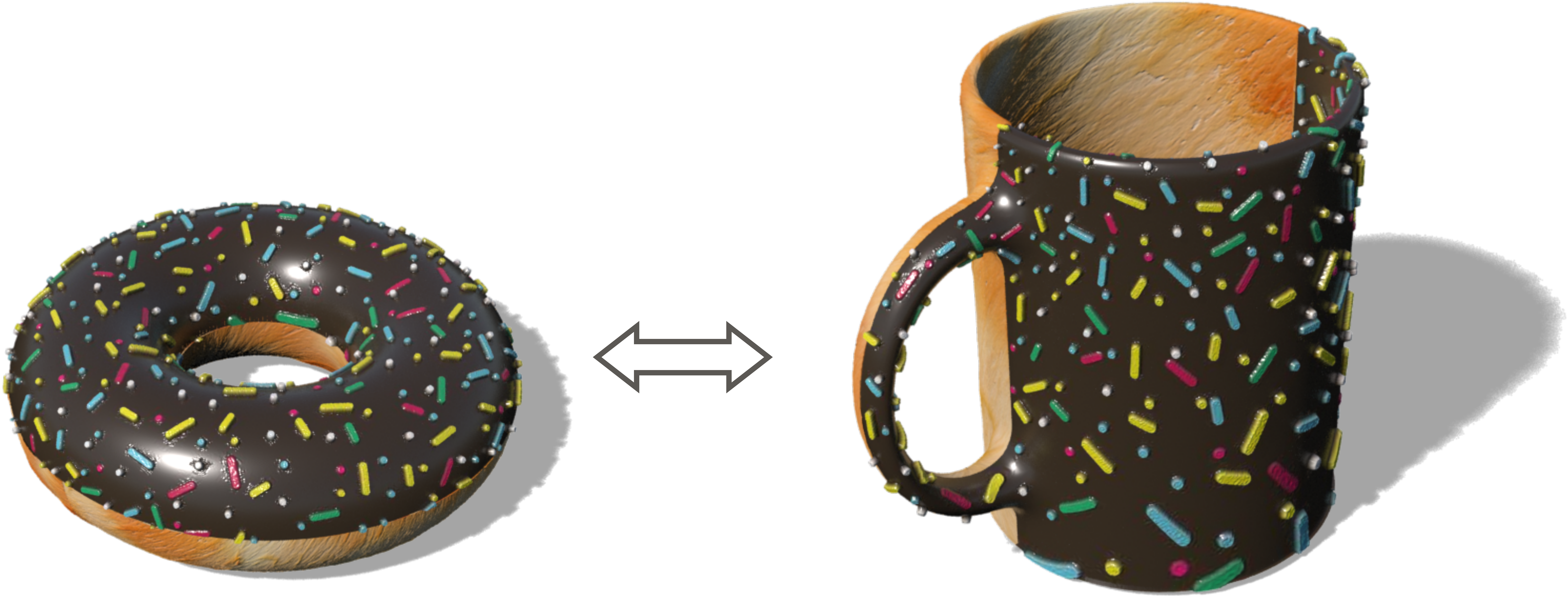
YES

How? And why do spinors work?

Immersion Theory of Surfaces

Topologist’s mug

Topologist’s mug



Topologist's mug

stay immersed throughout

regular homotopy

Topologist's mug

regular
homotopy

Topologist's mug

Topologist’s mug

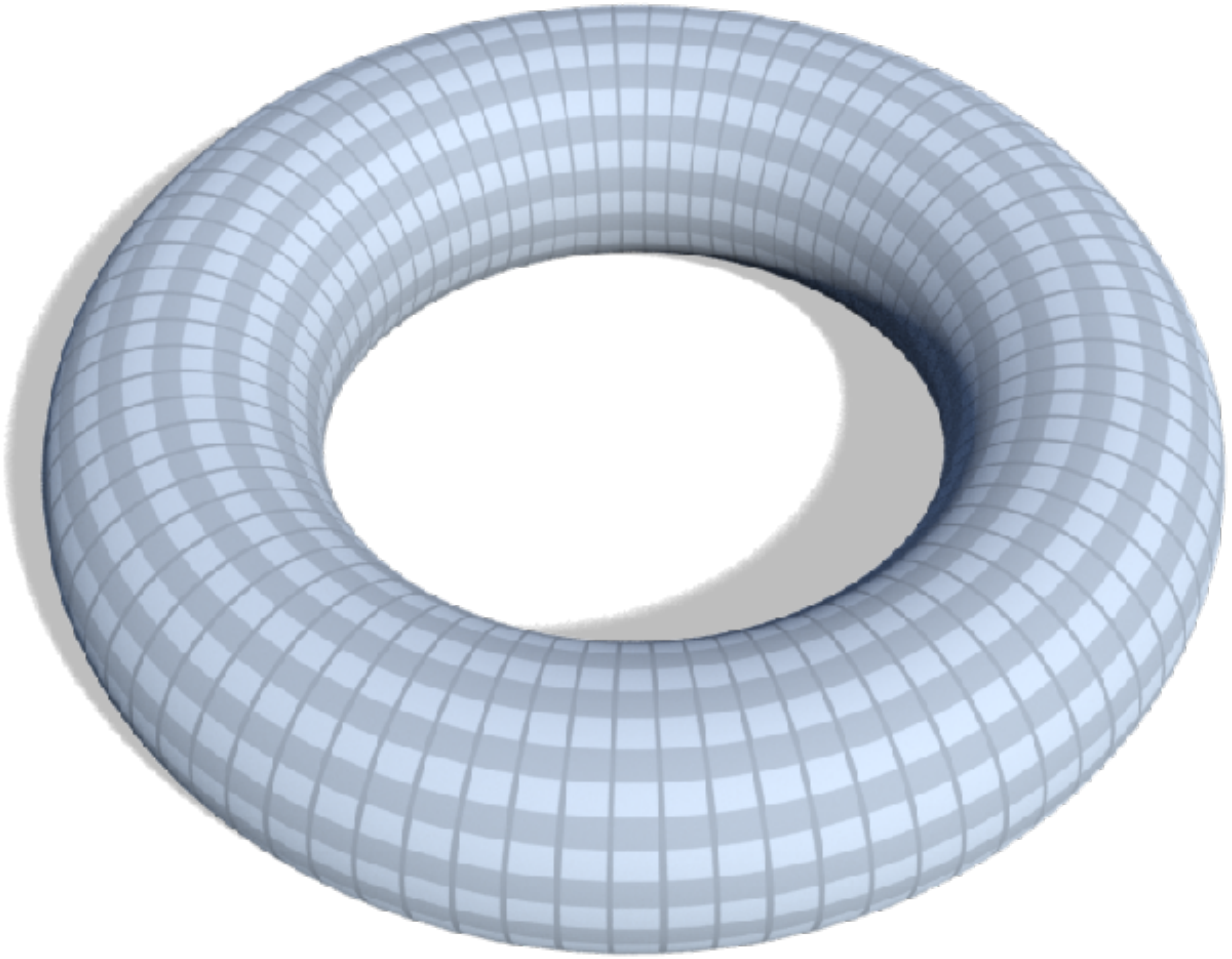
Topologist’s mug

~~regular homotopy~~

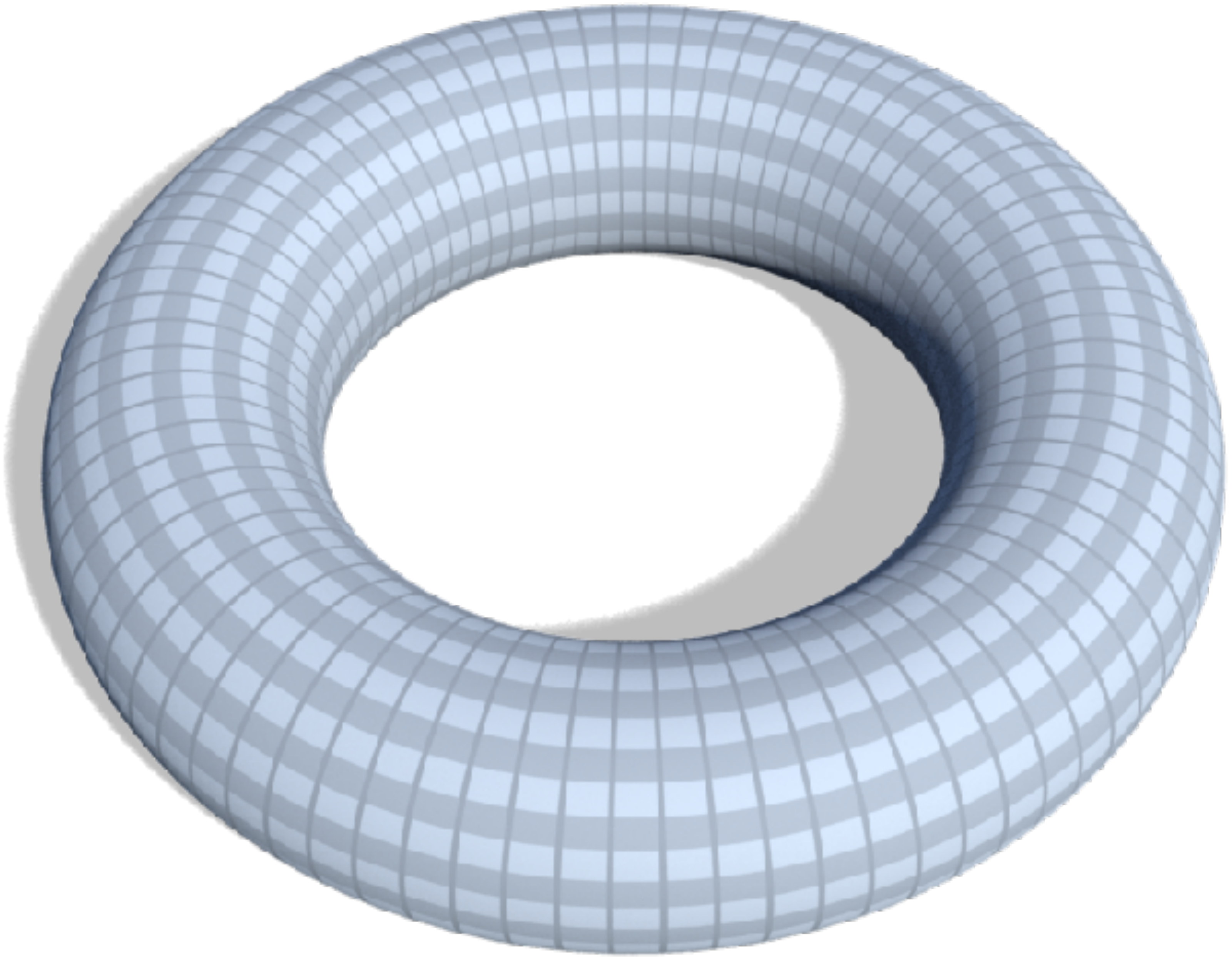
Regular homotopy classes

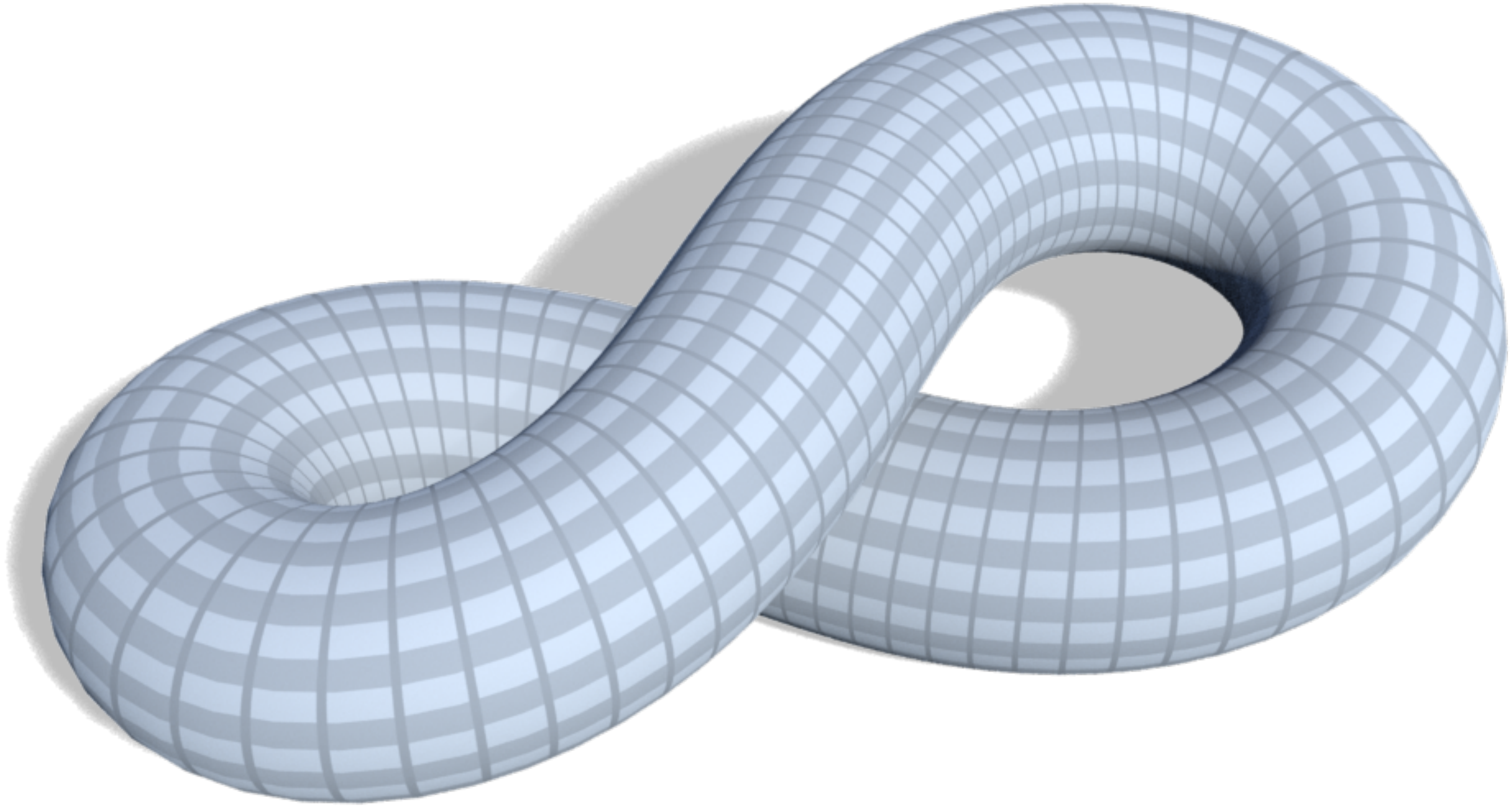
Regular homotopy class

Regular homotopy classes

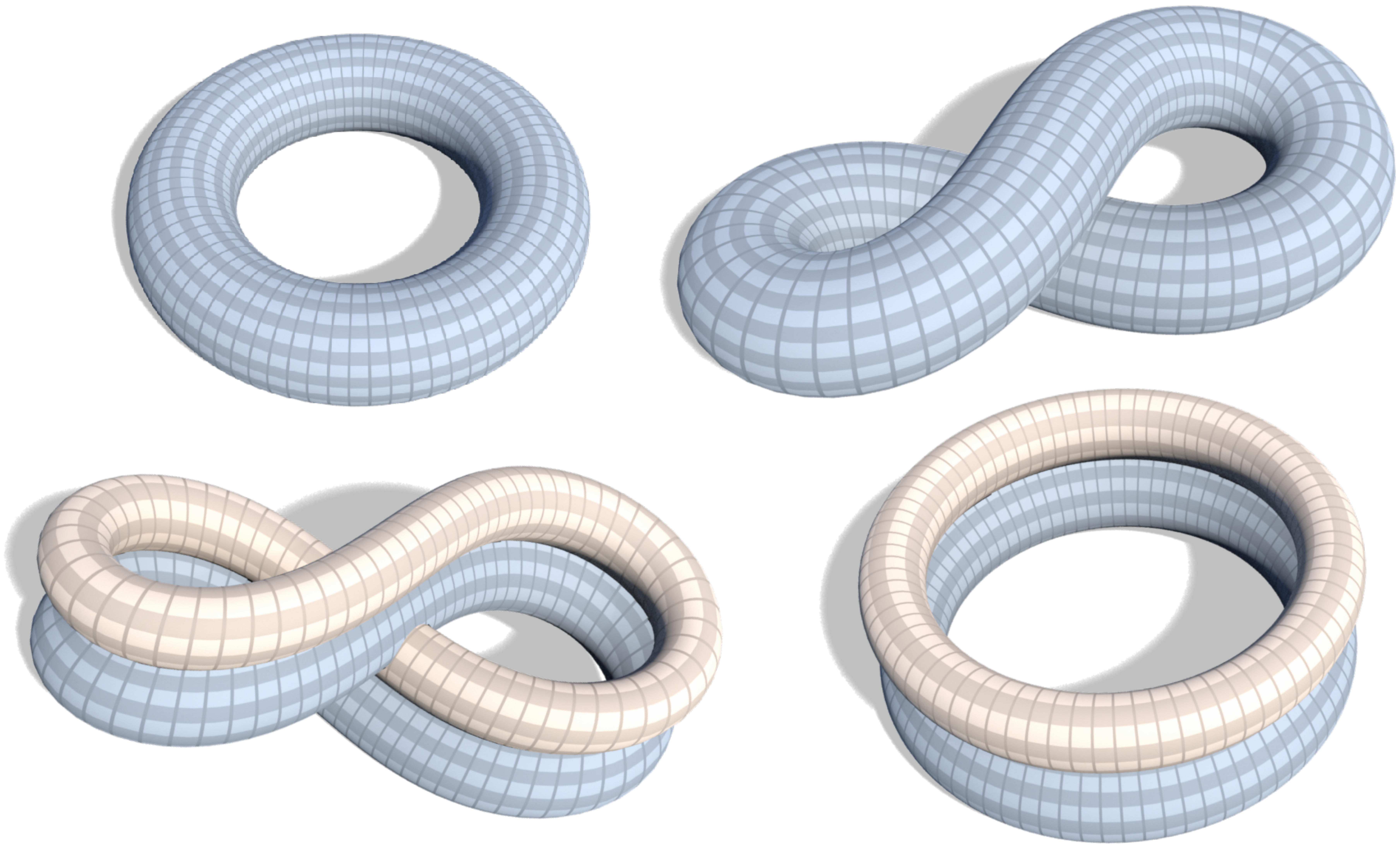


Regular homotopy classes





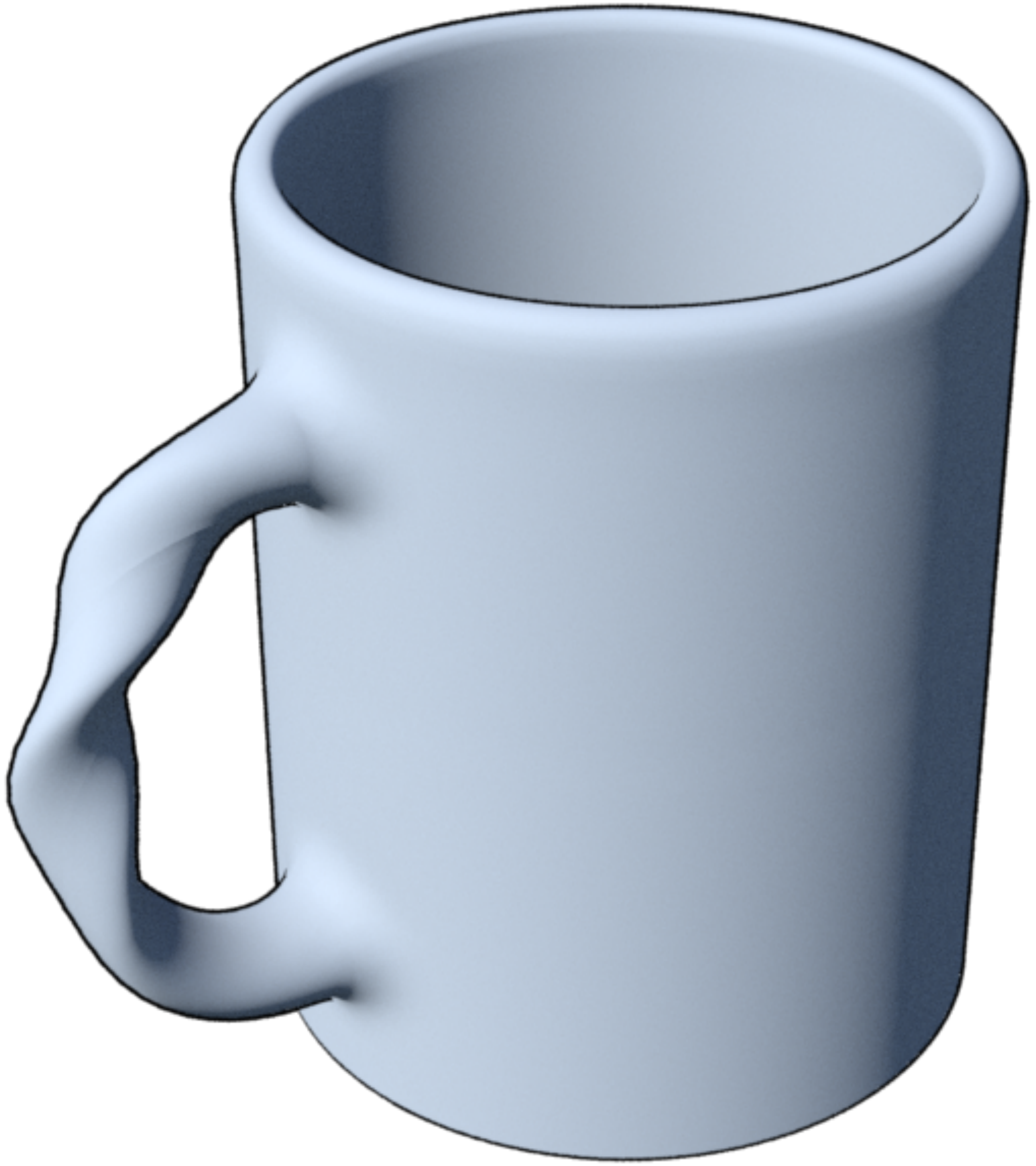
Regular homotopy classes



Closed strips

Immersion?

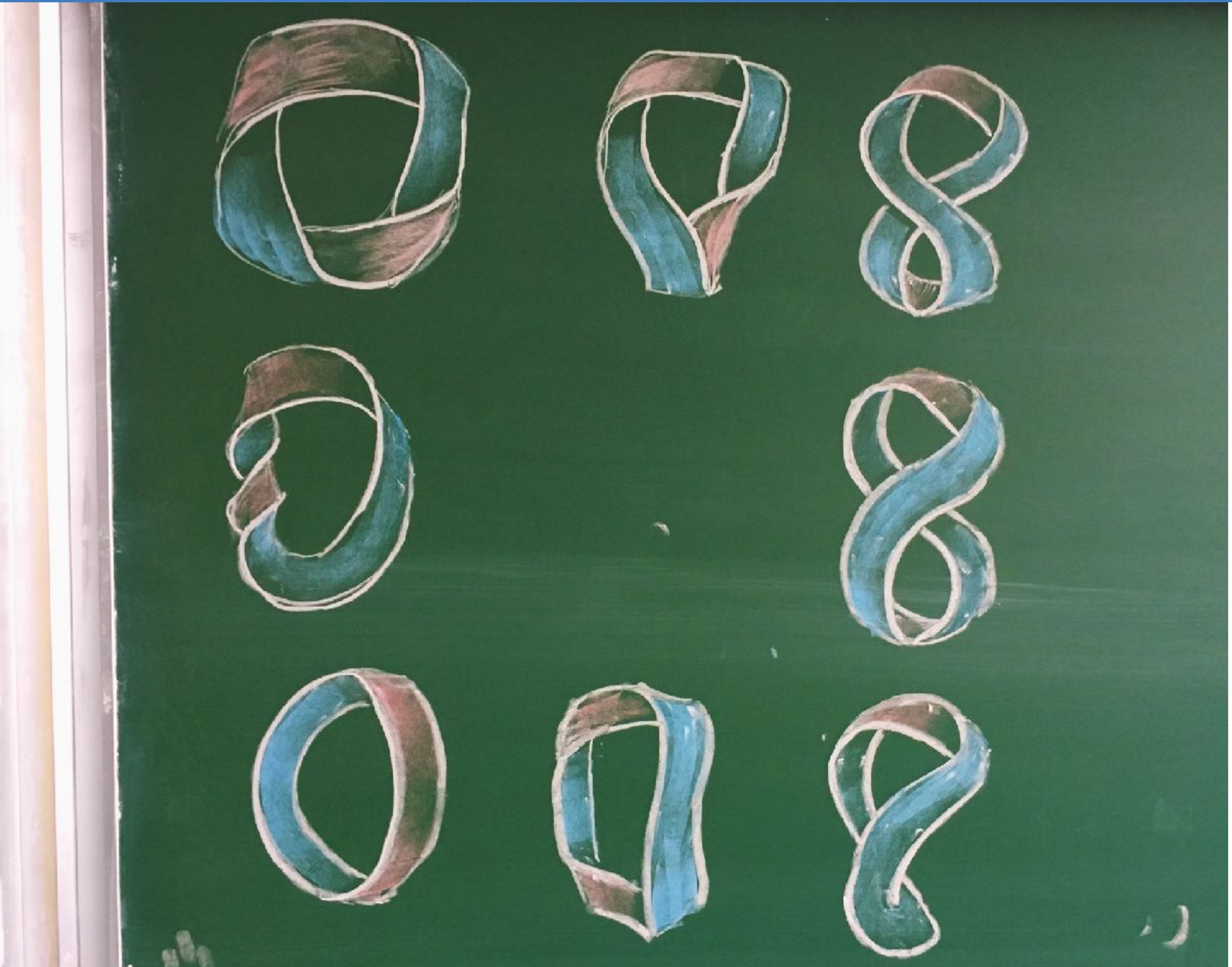
Regular homotopy class?



Closed strips

closed strip

Closed strips



Closed strips

Theorem (Closed strips)

There are 2 regular homotopy classes for oriented closed strips.

Closed strips

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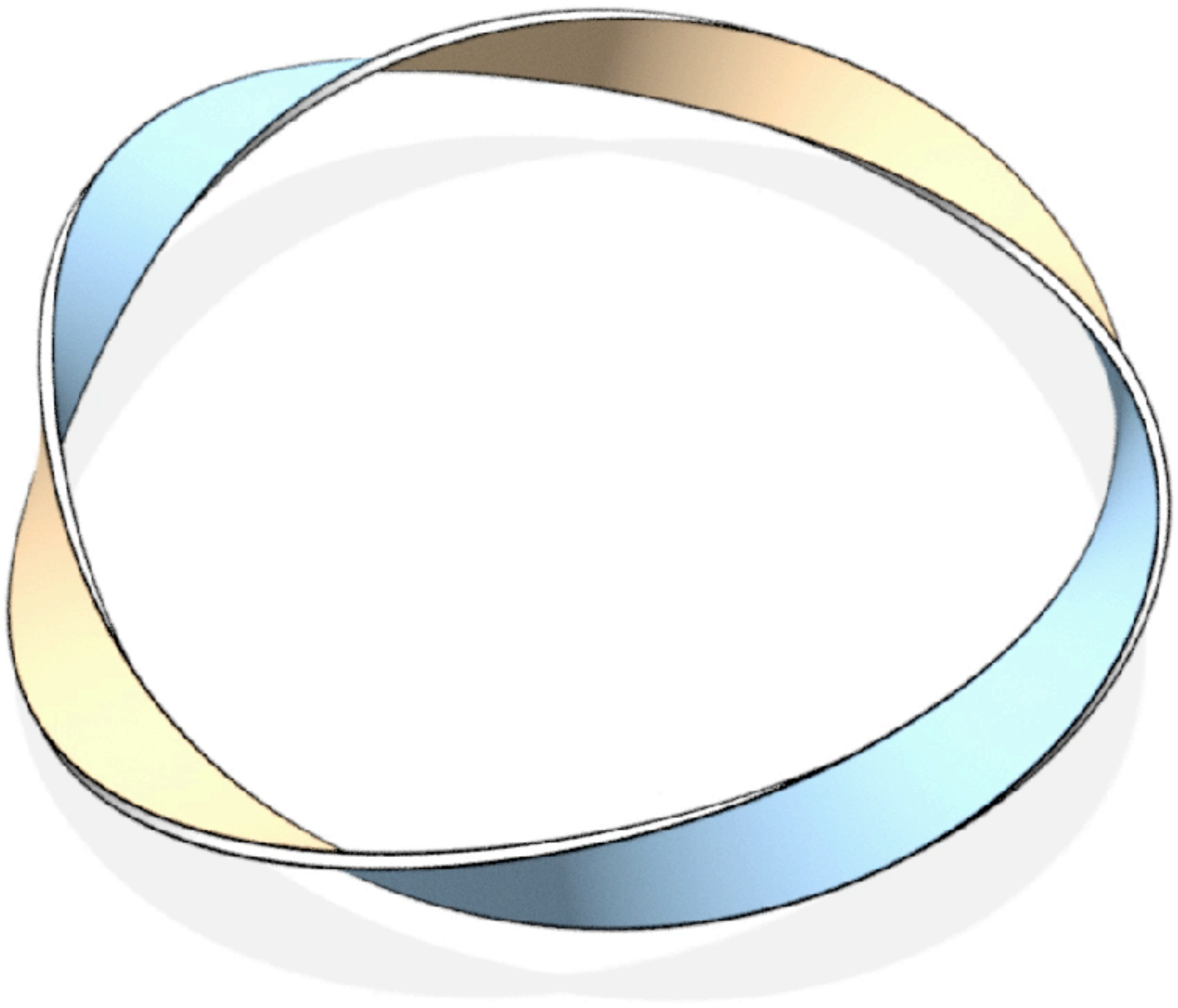

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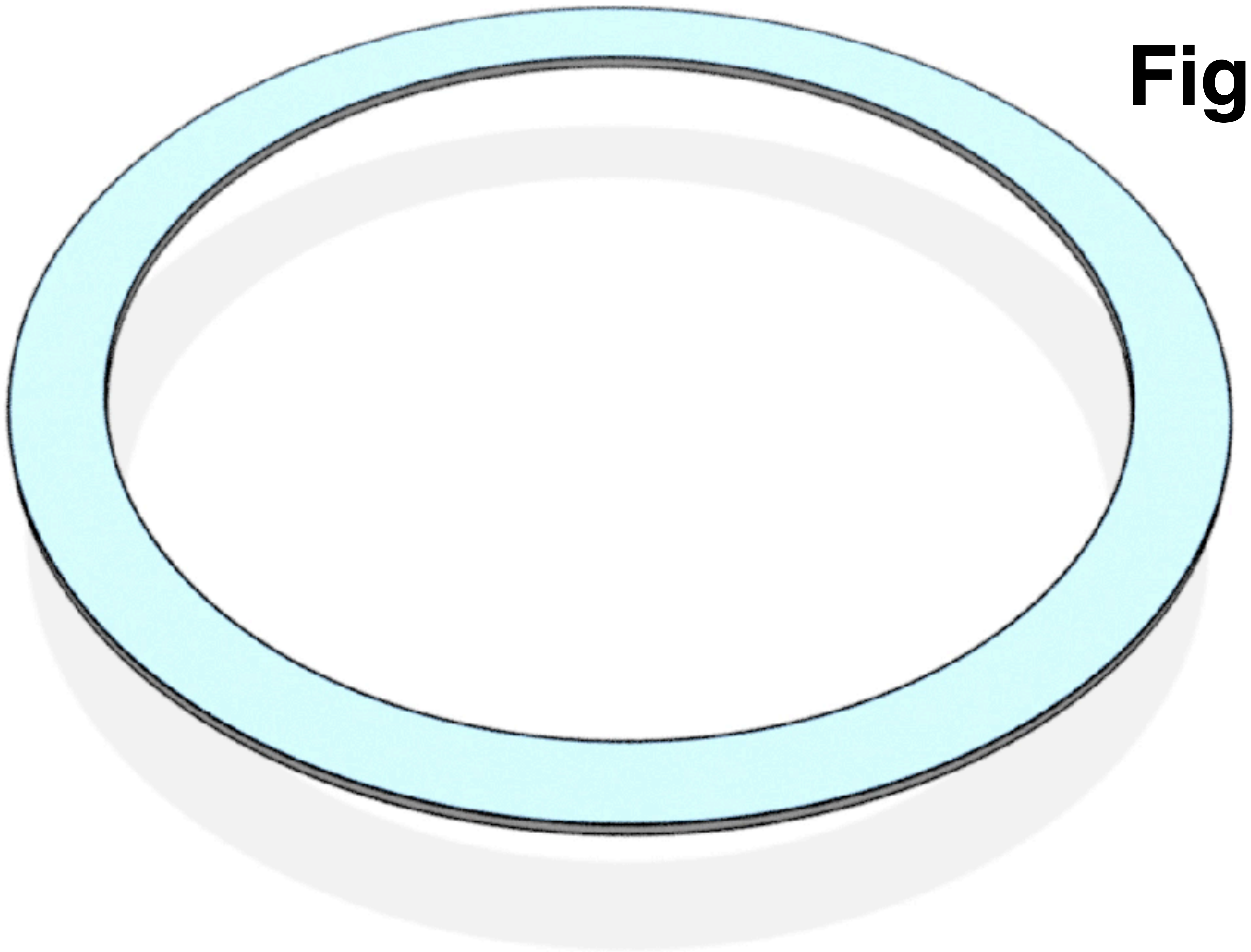


Figure-0

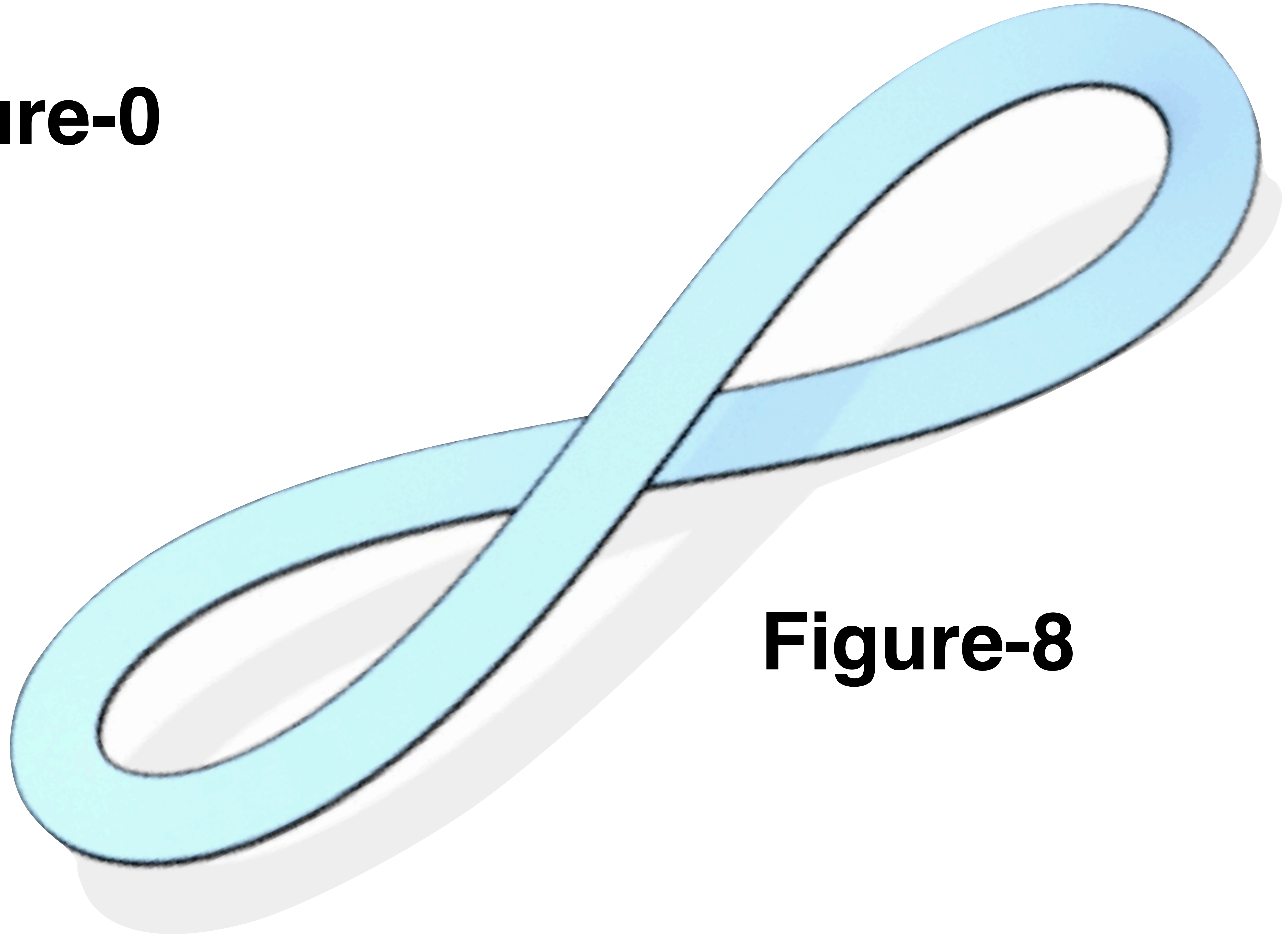
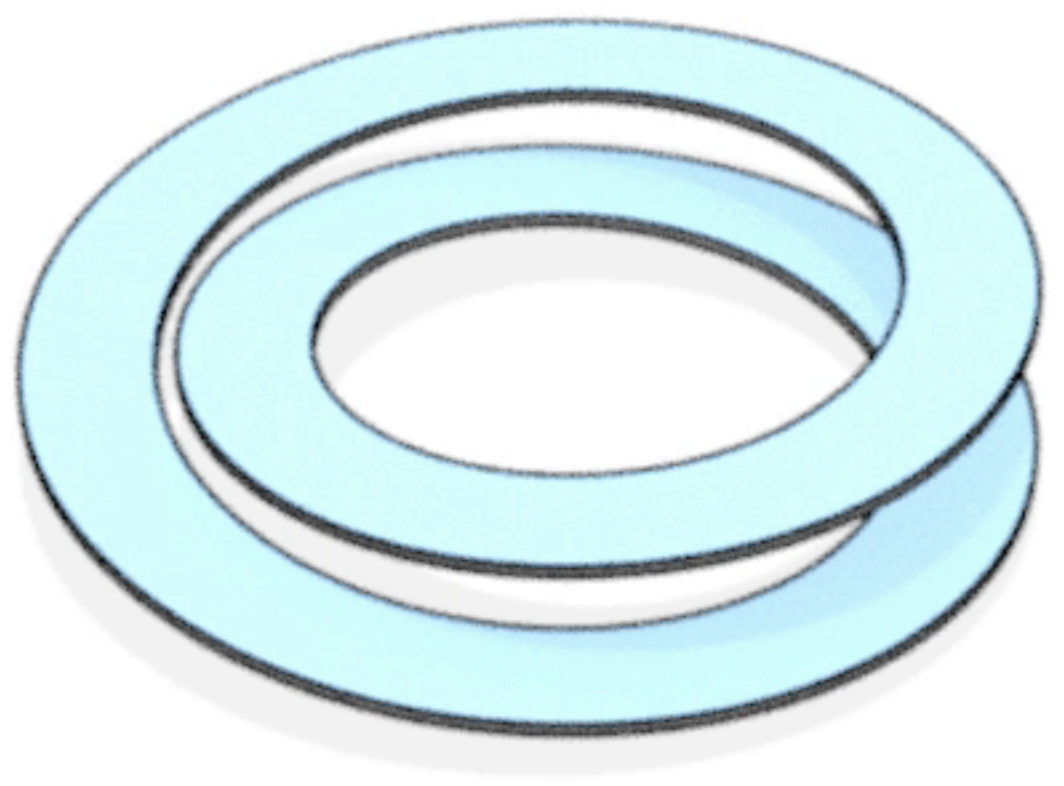


Figure-8

Closed strips



Immersibility of disks

Theorem (Immersibility of Disks)

A disk can be perturbed into an immersion if and only if its boundary strip is a Figure-0.

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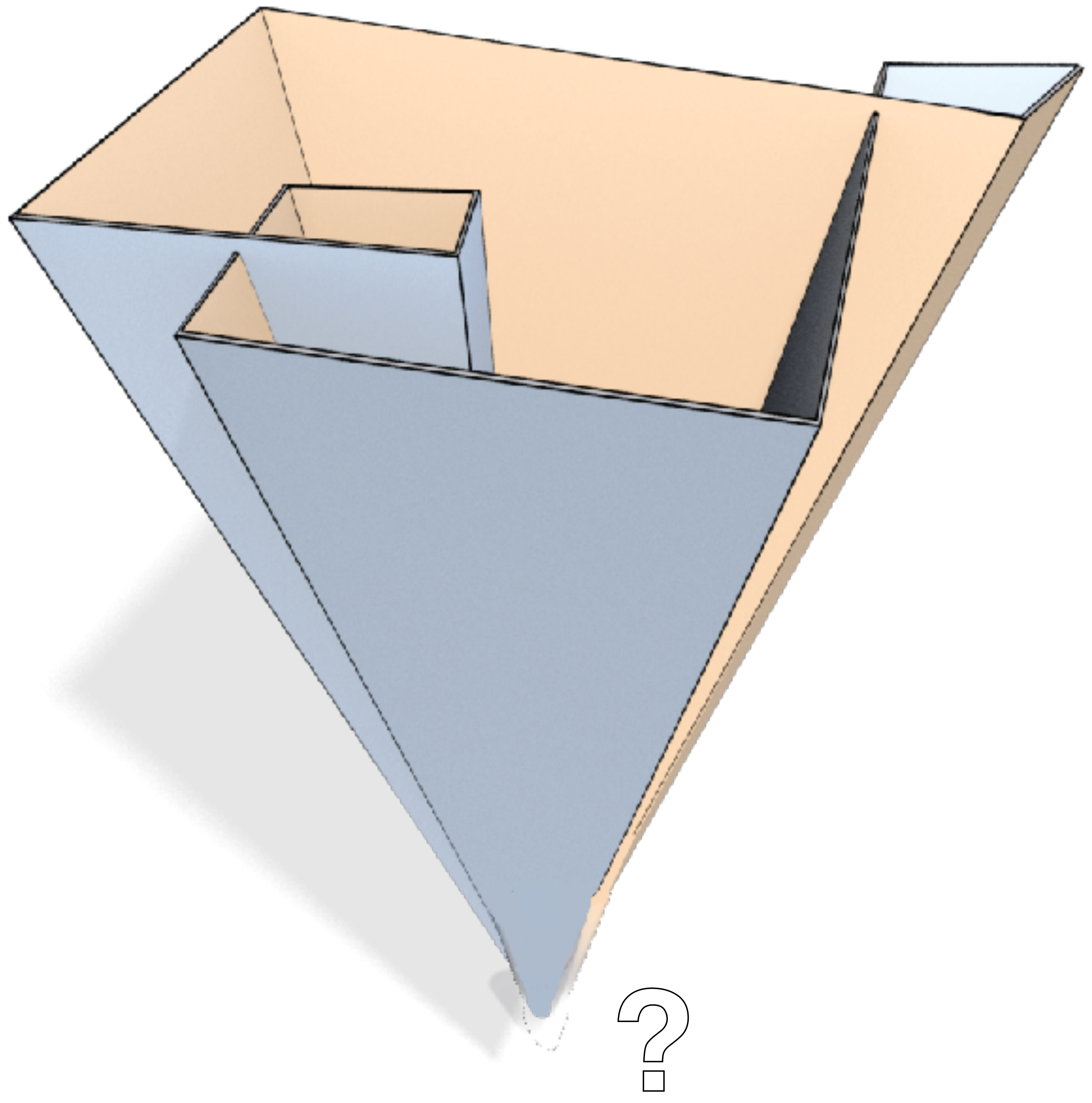

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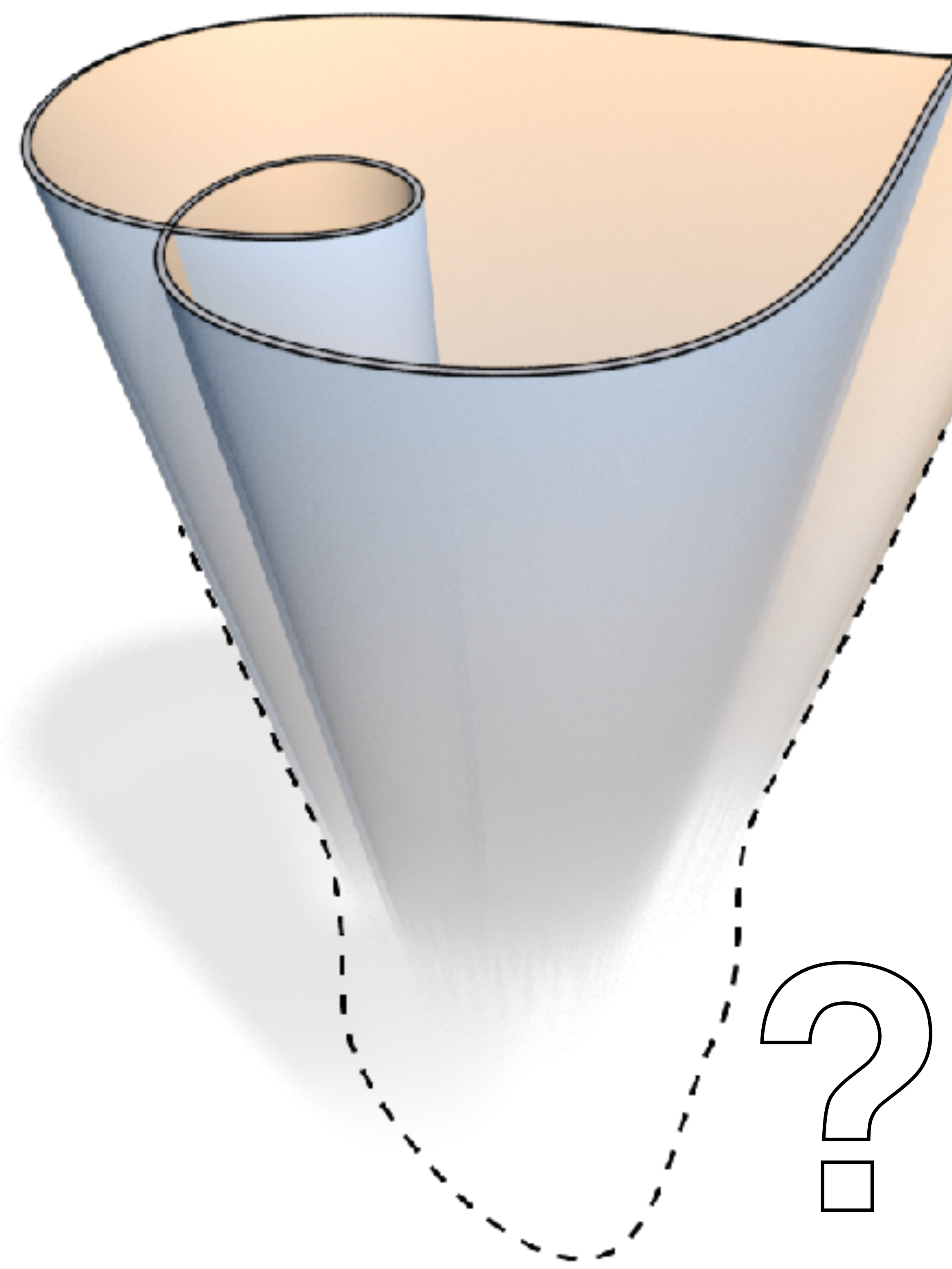

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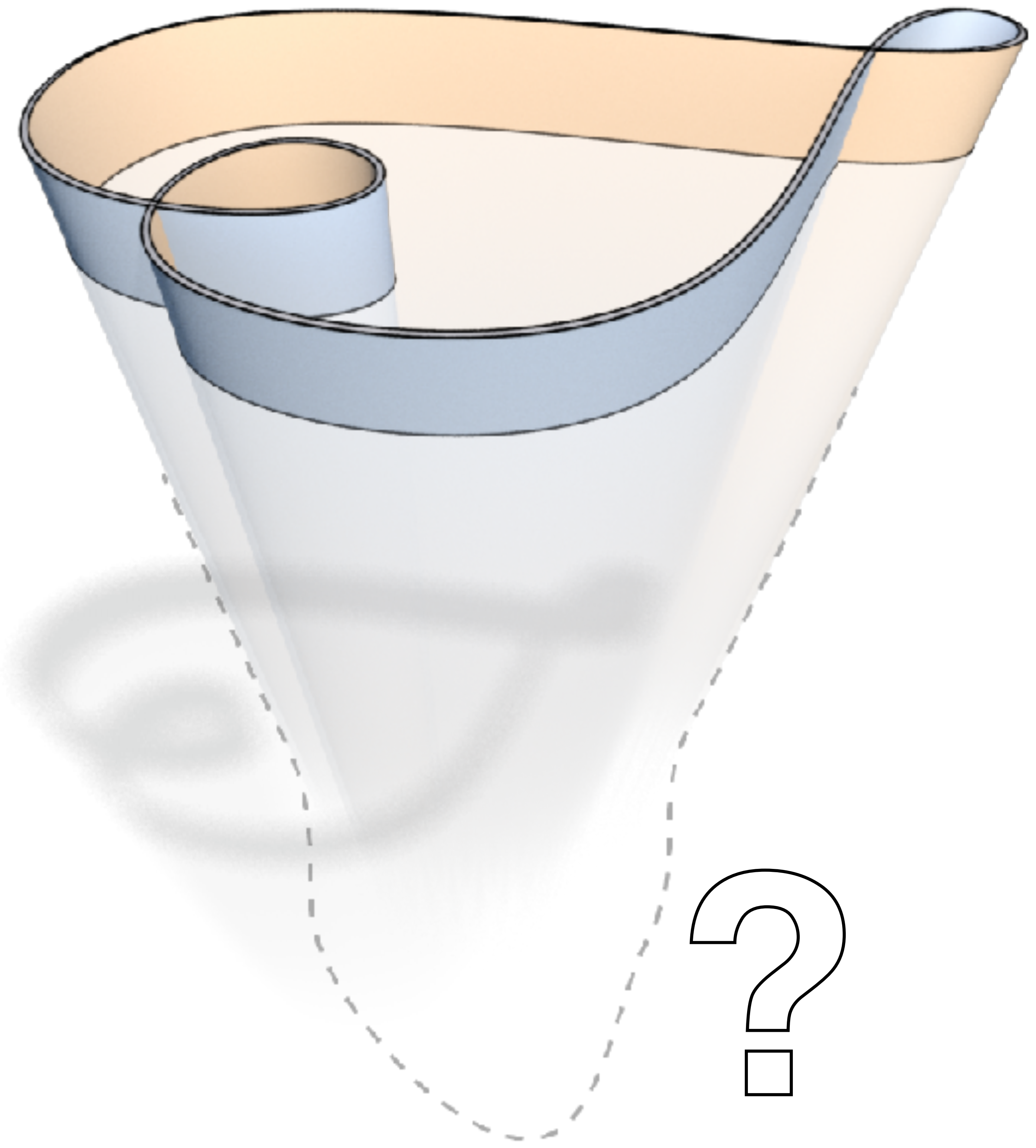

Figure-0

Immersibility of disks

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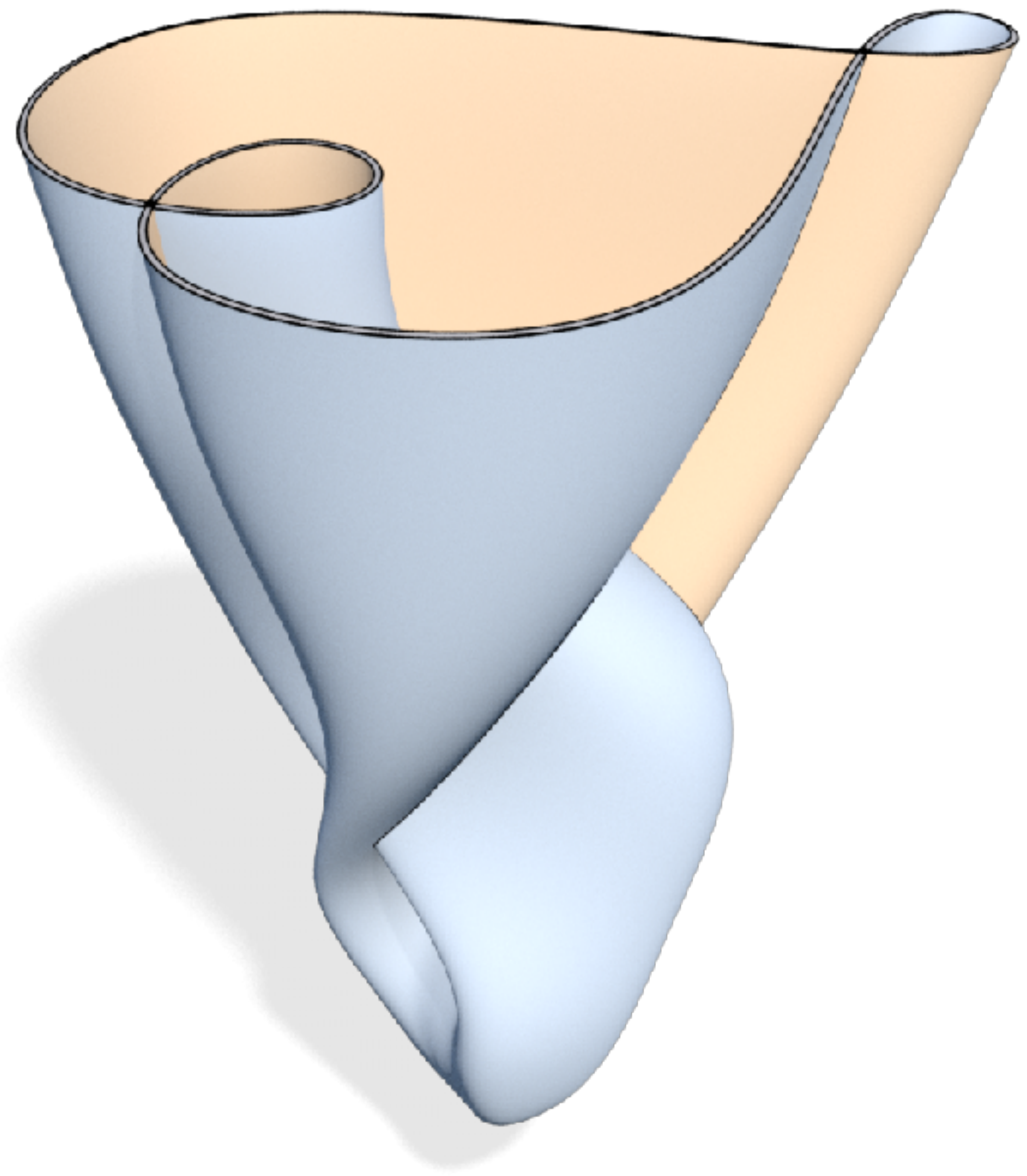

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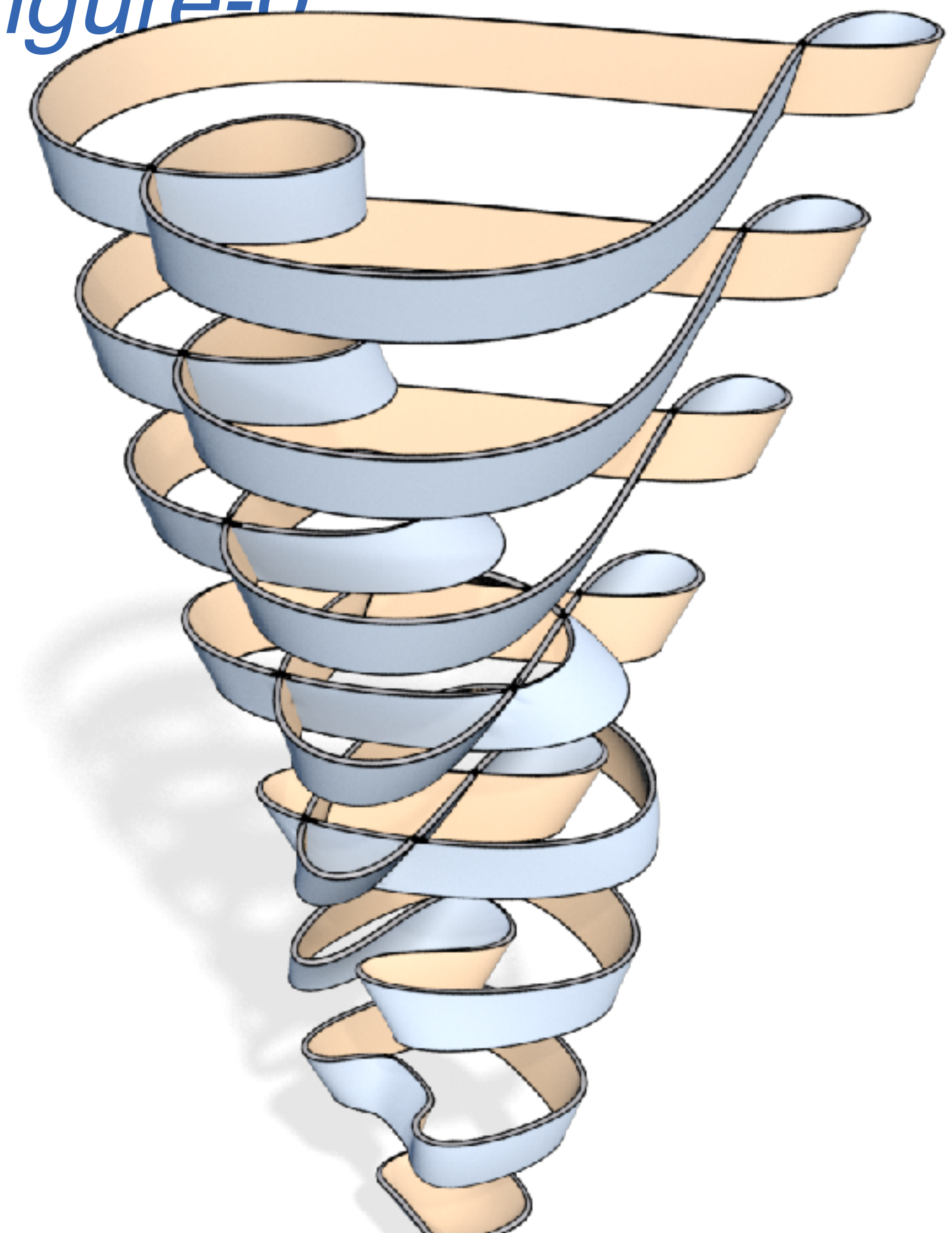

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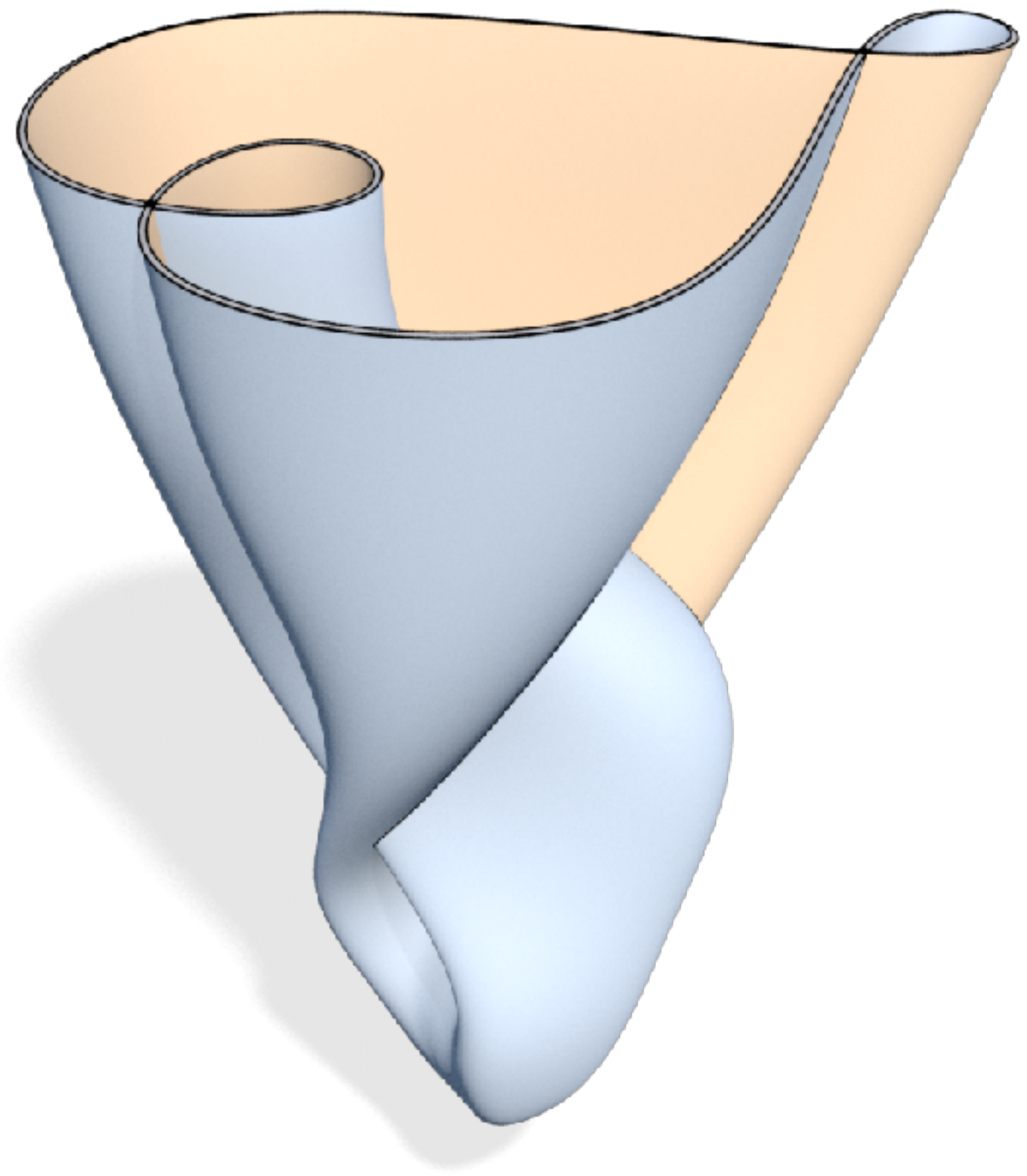

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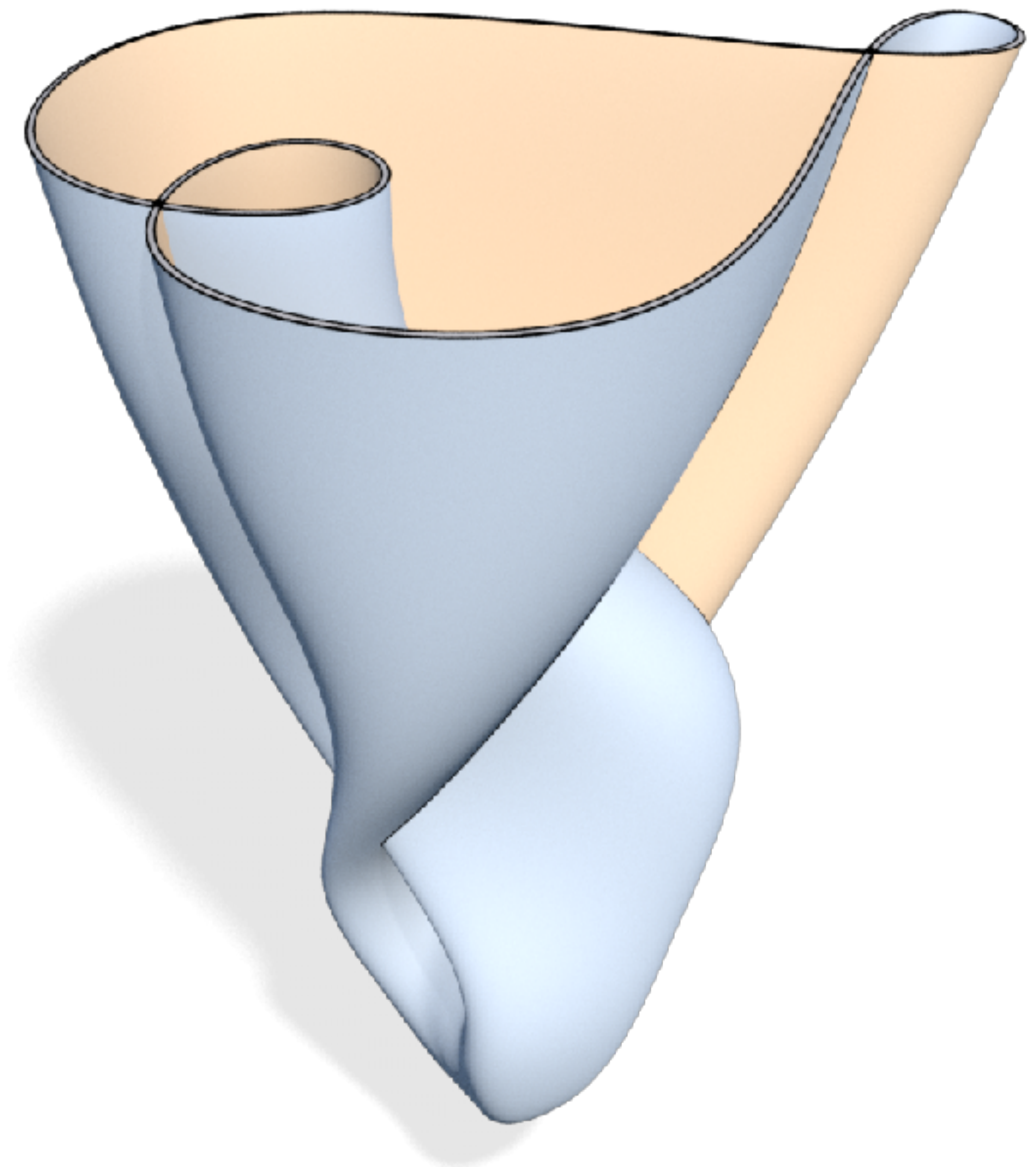


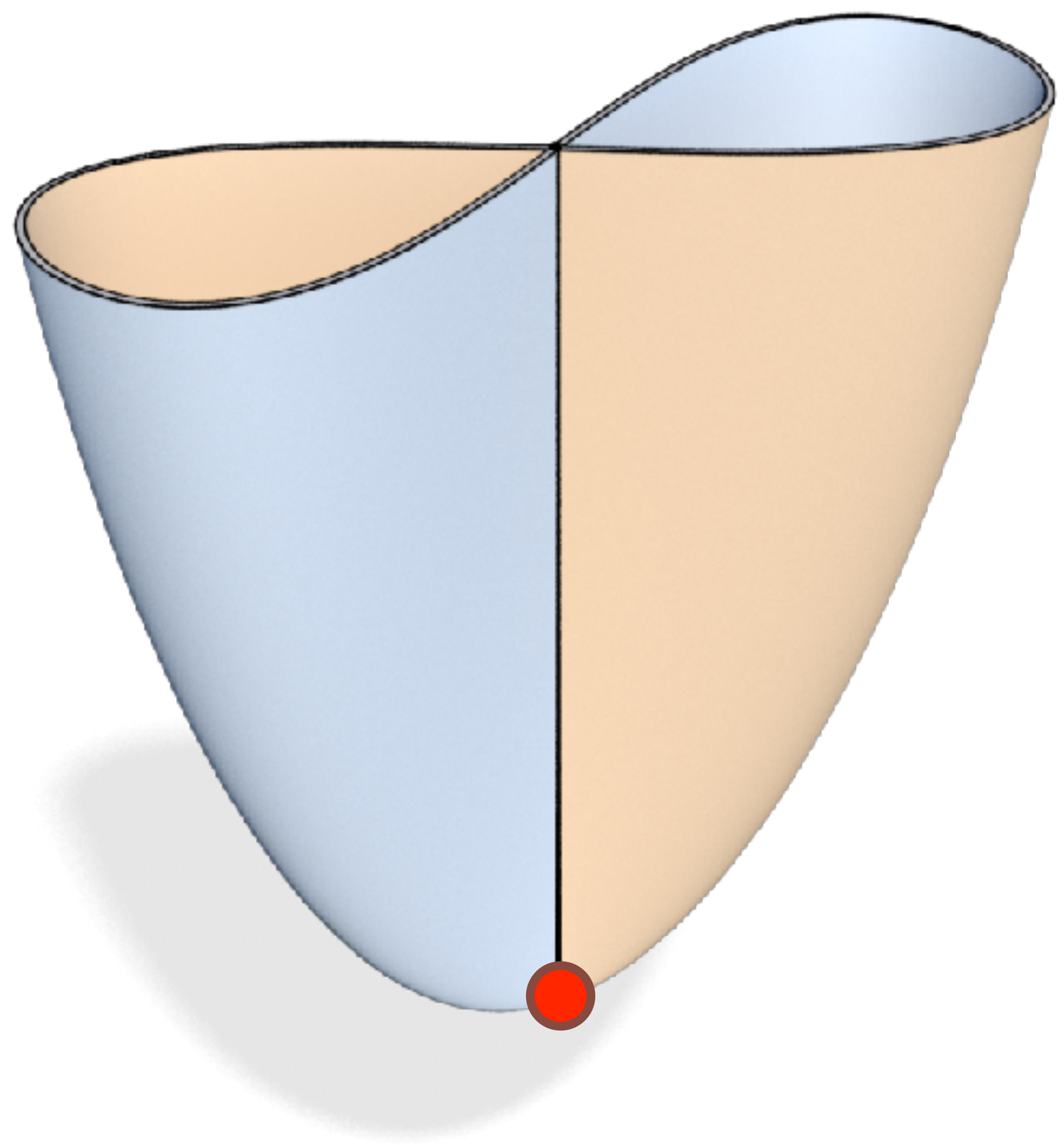
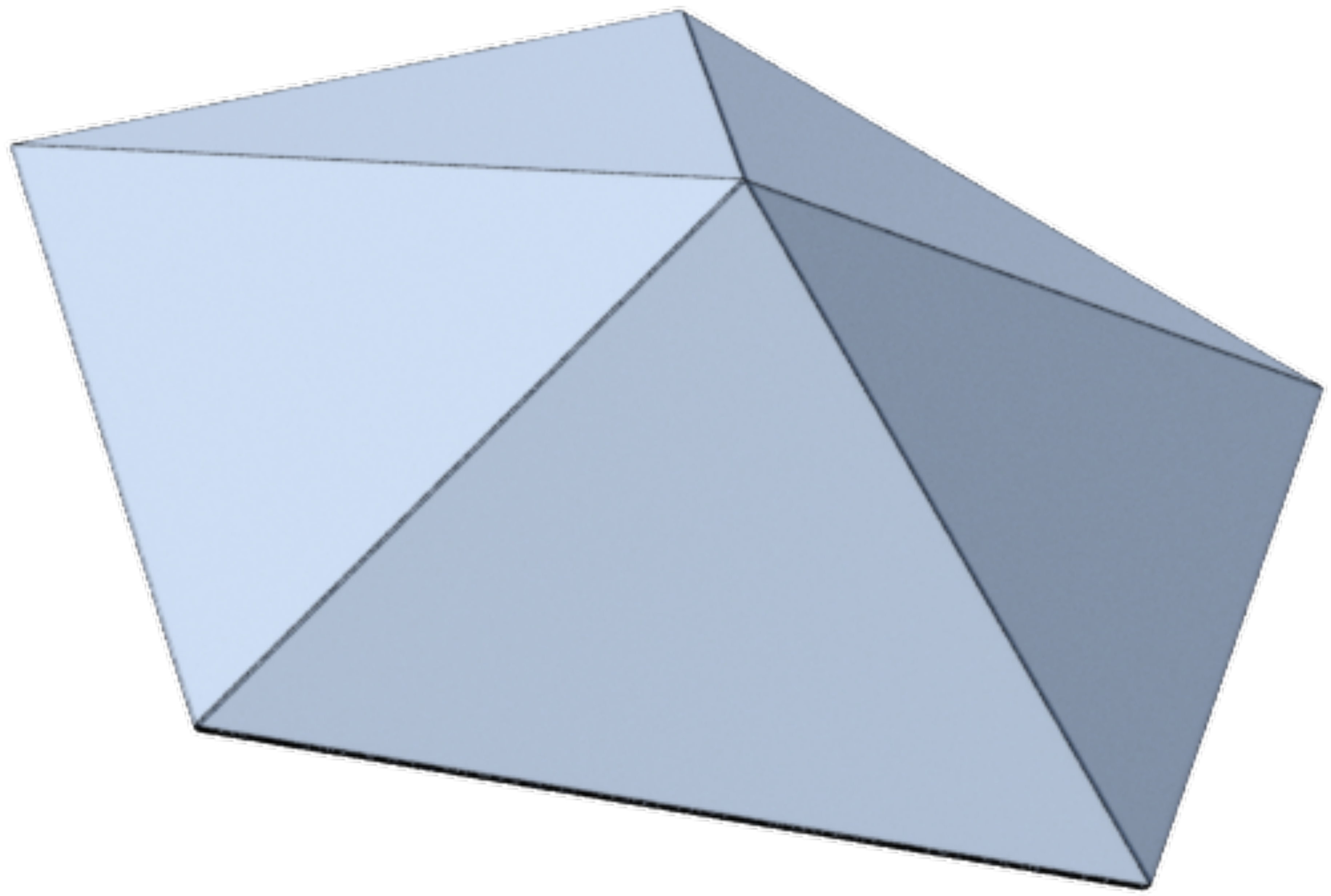


Figure-8

$\Downarrow$

pinch point

Immersion condition



Immersion condition

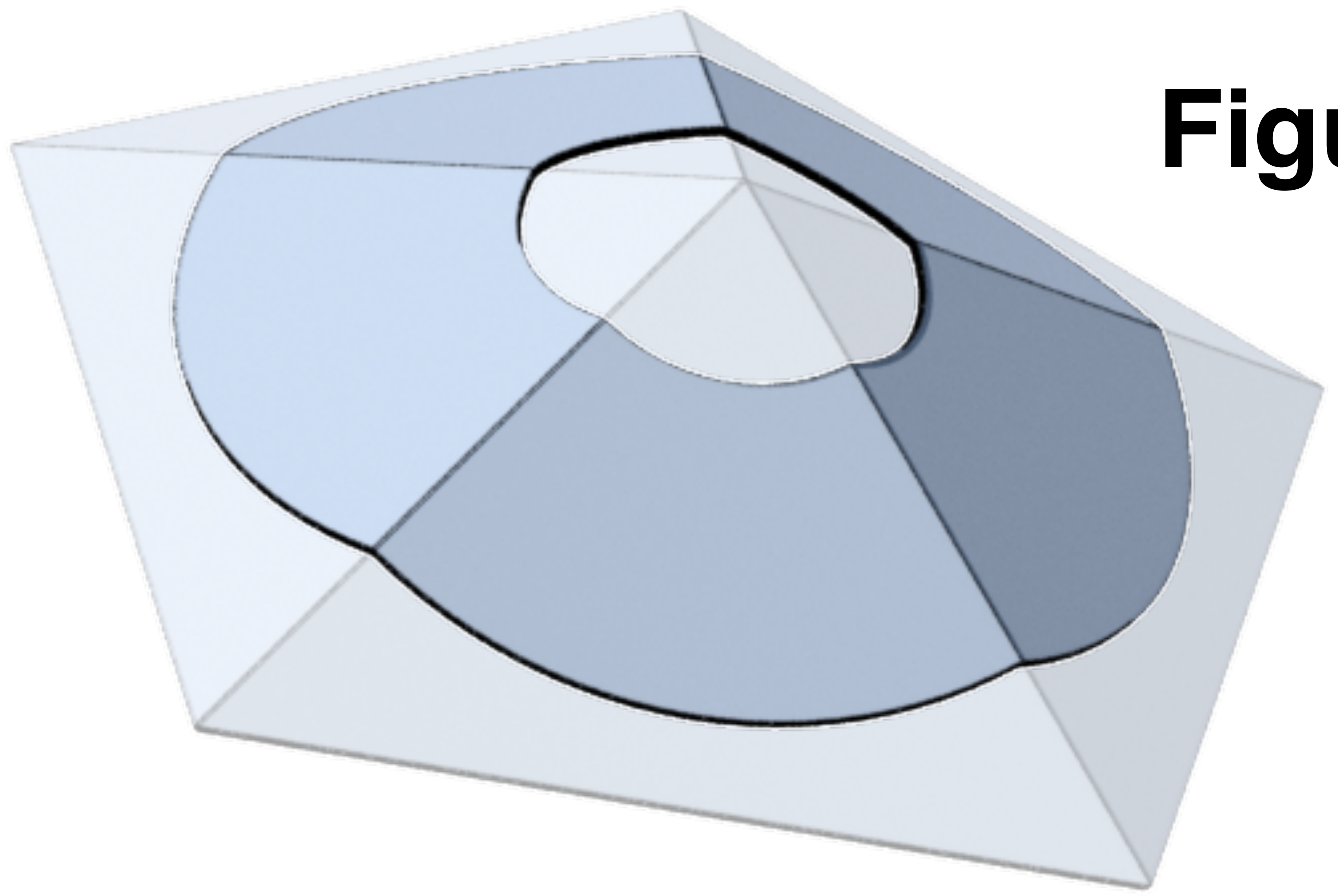


Figure-0

Immersion condition

Definition

A vertex is said to be **almost immersed** if its one-ring triangle strip is a Figure-0.

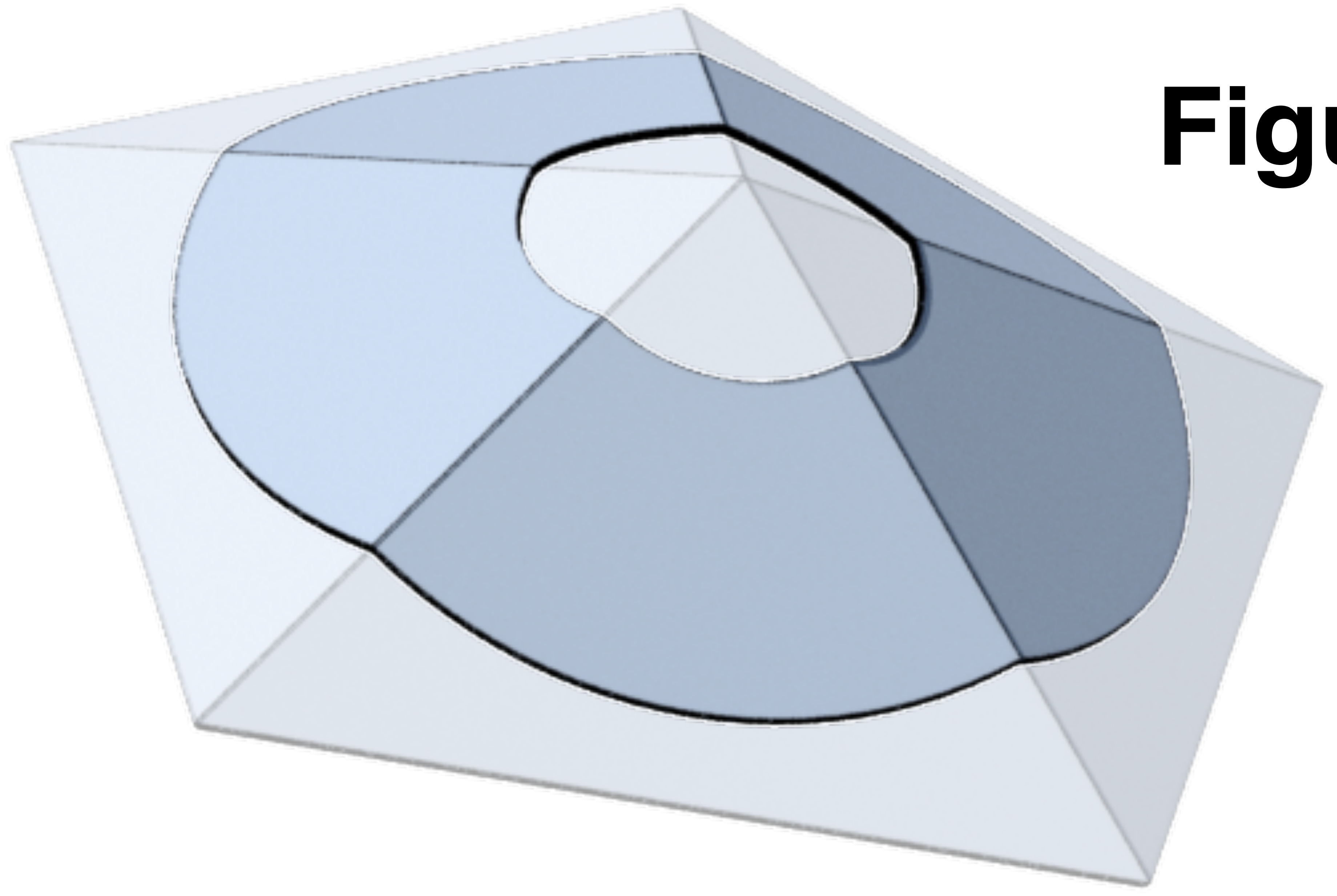


Figure-0

Immersion condition

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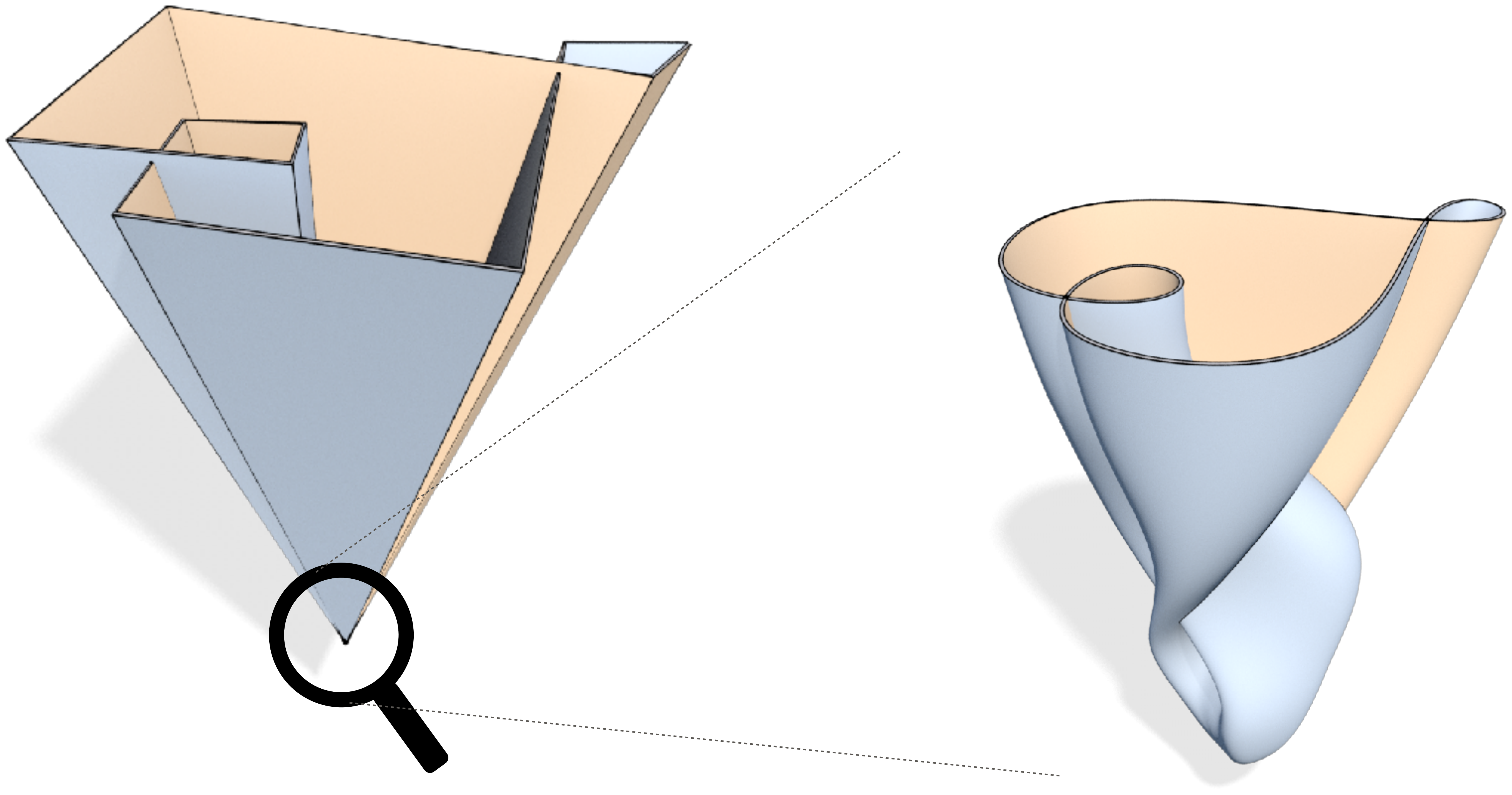


Immersion condition

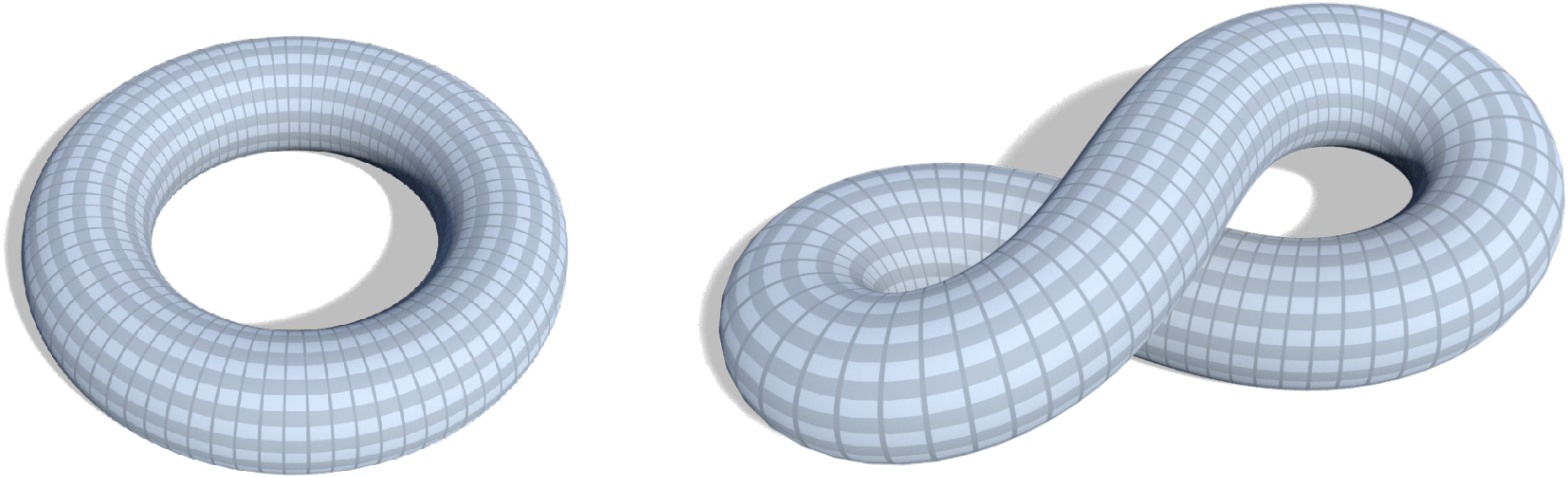
Definition

A vertex is said to be **almost immersed** if its one-ring triangle strip is a Figure-0.

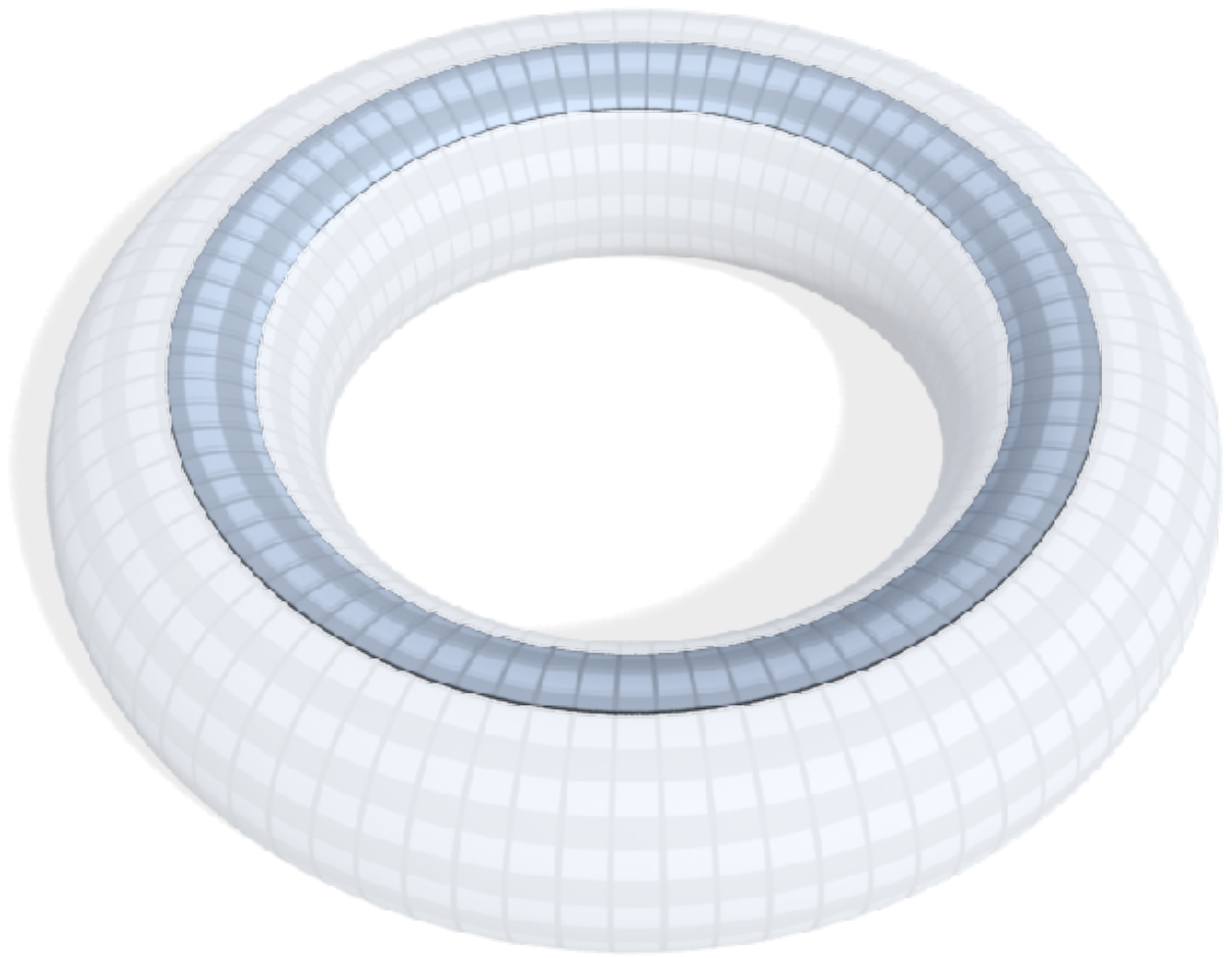
Definition

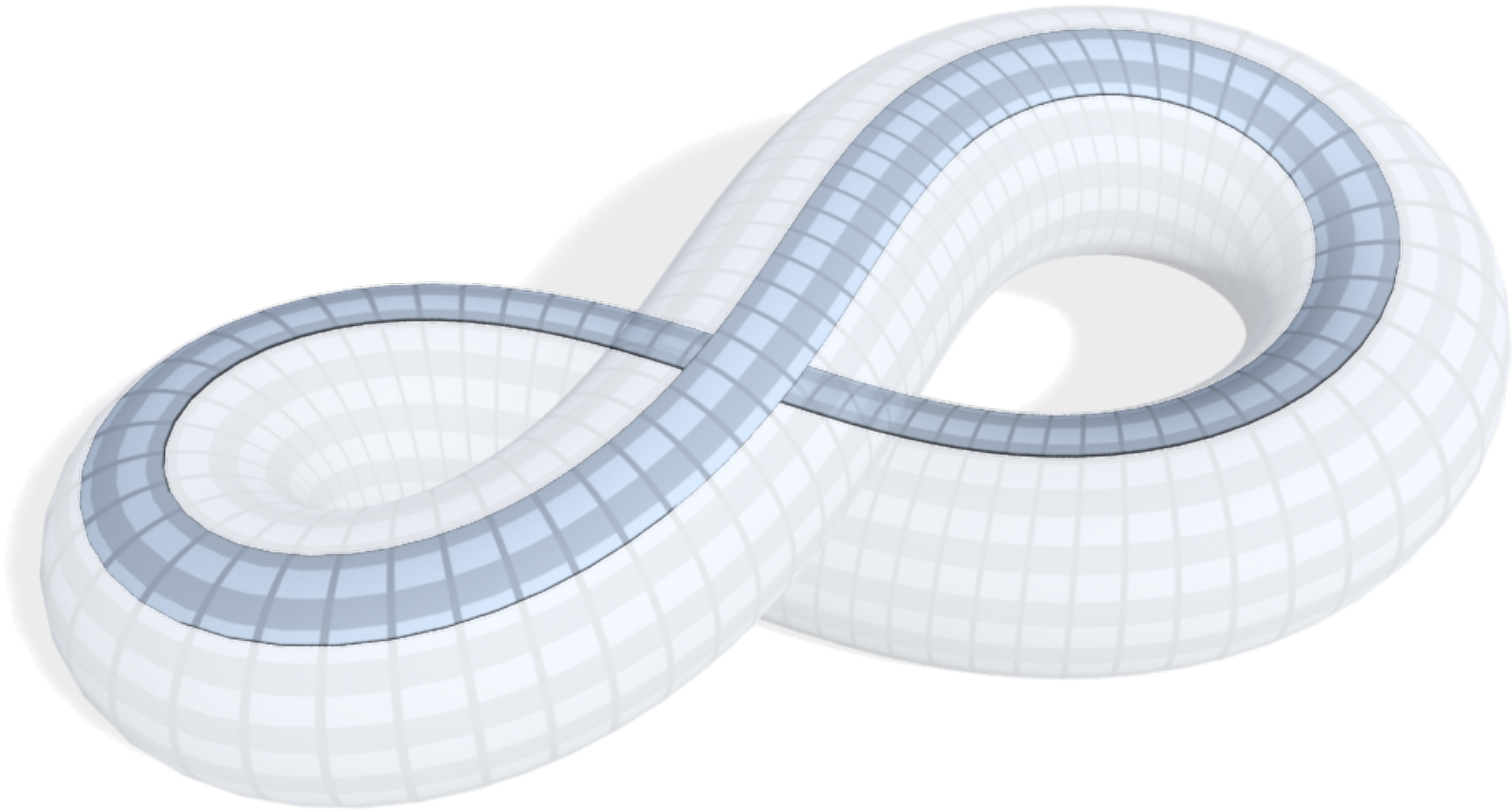
A simplicial surface is **almost immersed** if all vertices are almost immersed. That is, all contractable strips are Figure-0.

Global strips



Global strips

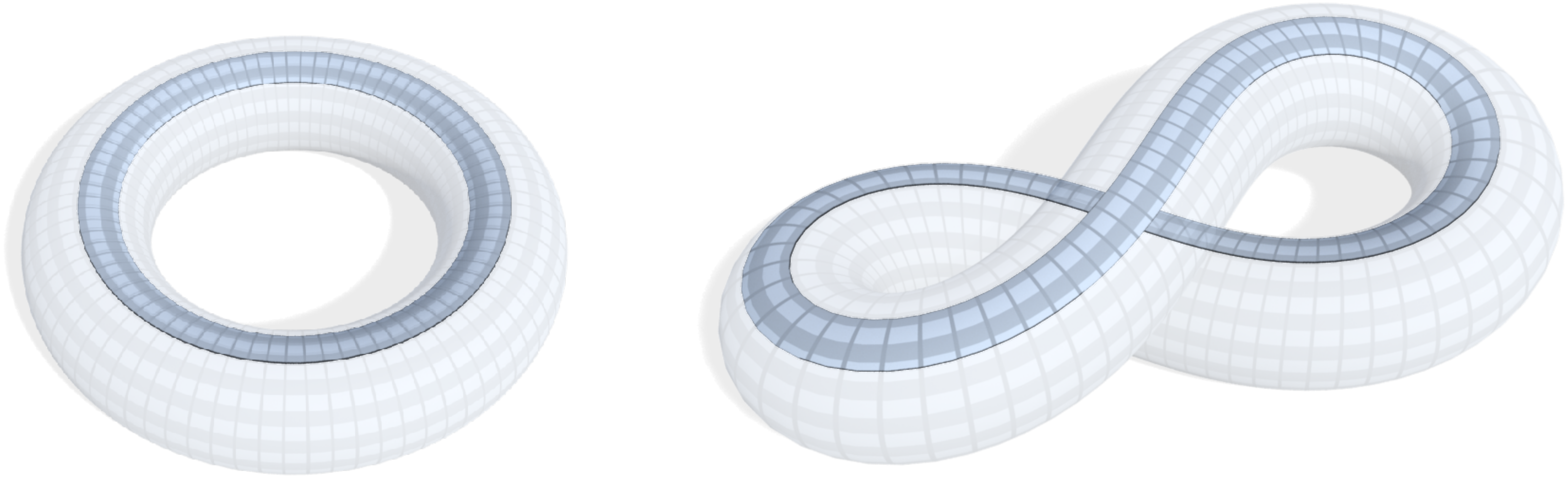




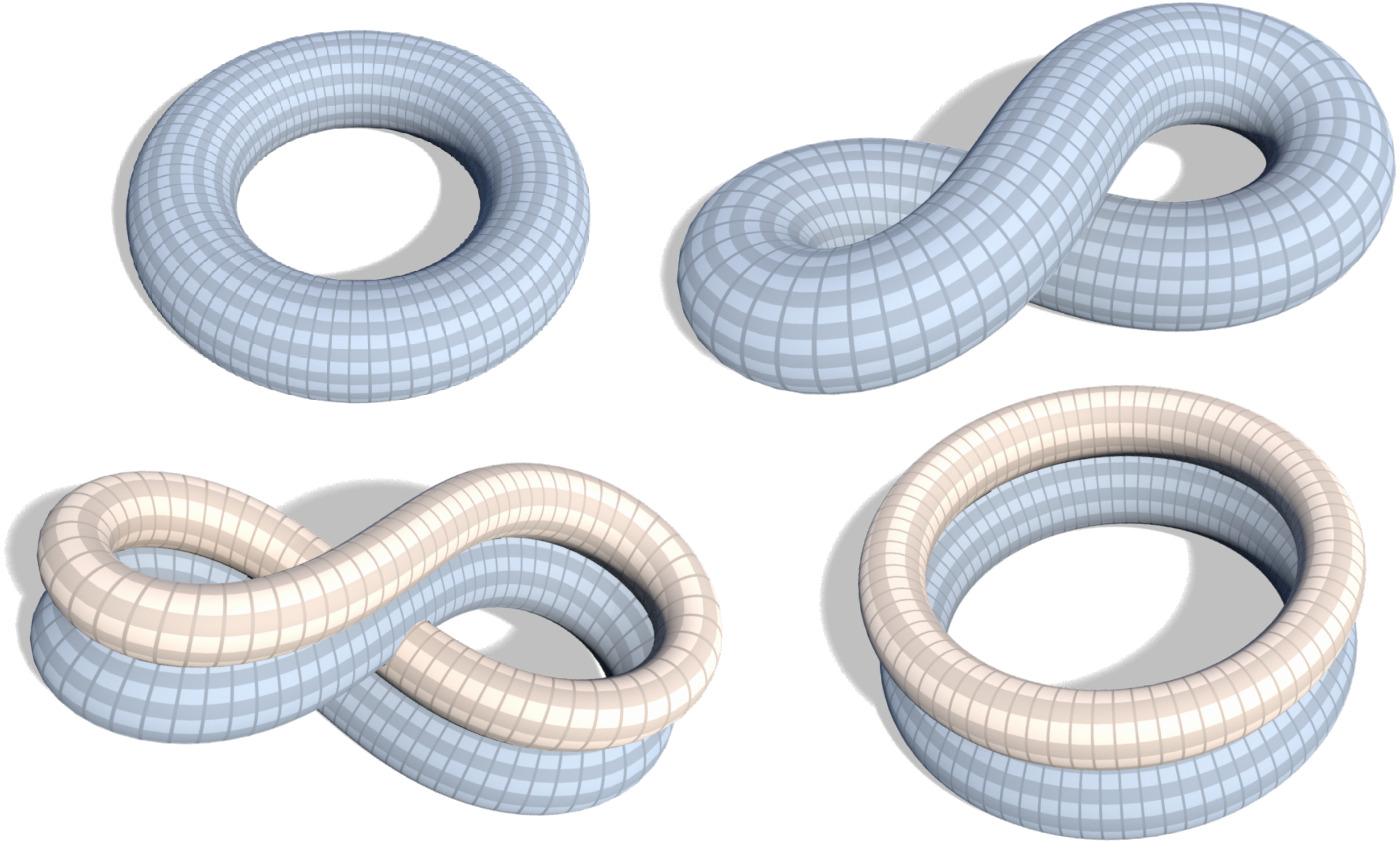
Global strips

Theorem (Regular homotopy)

Two immersions are regular homotopic if and only if their global strips share the same Figure-8/0 type.



Global strips



Original question

*Can we construct surfaces that are guaranteed to be **immersions**?*

Original question

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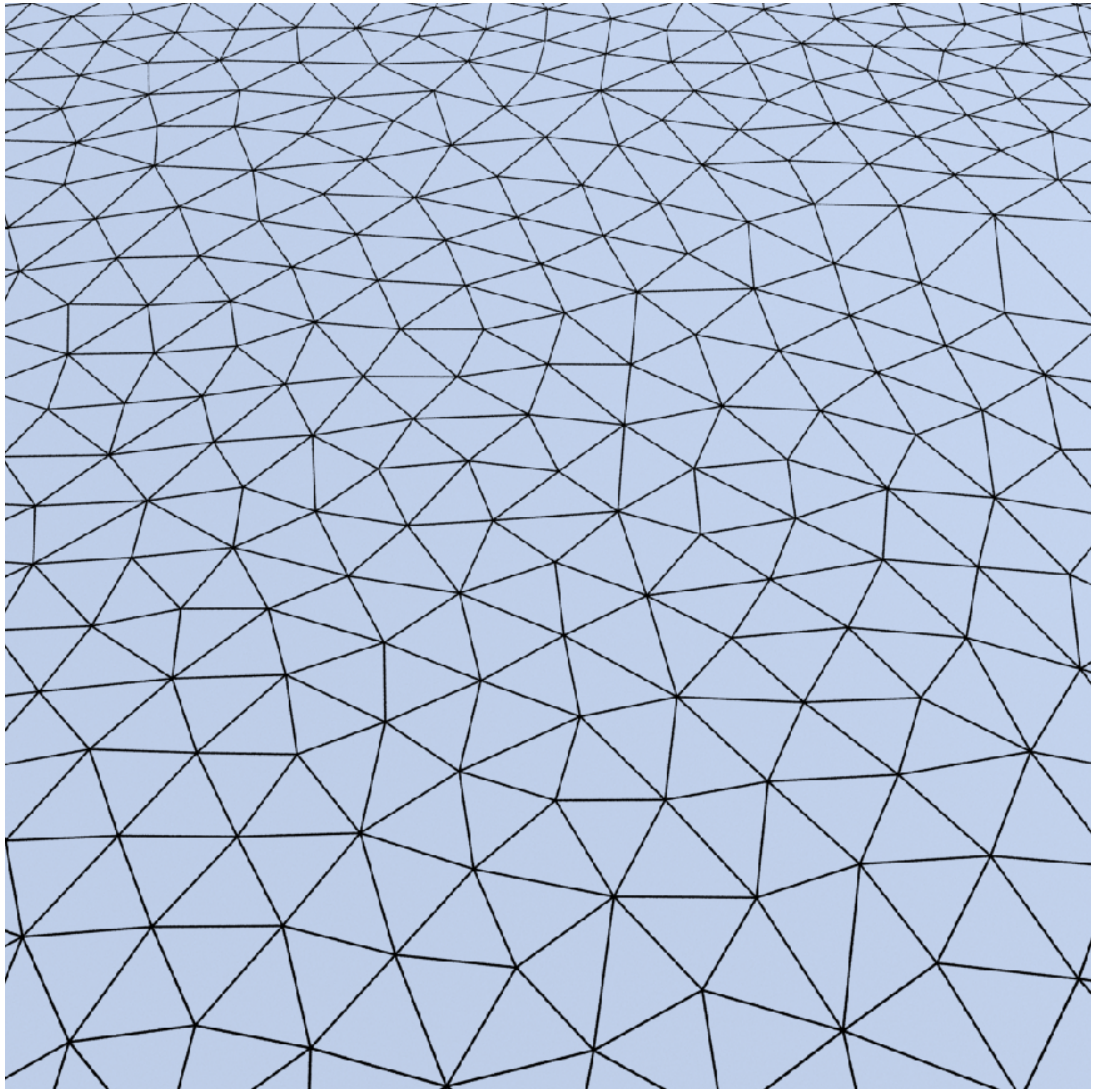
Can we “control” the Figure-8/0 type of all strips?

Original question

Can we "control" the Figure-8/0 type of all strips?

- ***Algebraic description*** of the strip types.
- "***Rims***" measure deviation from the desired strip configuration.
- Encode the above algebraic objective in the ***gauge field*** for the ***spinors***.

The space of closed strips



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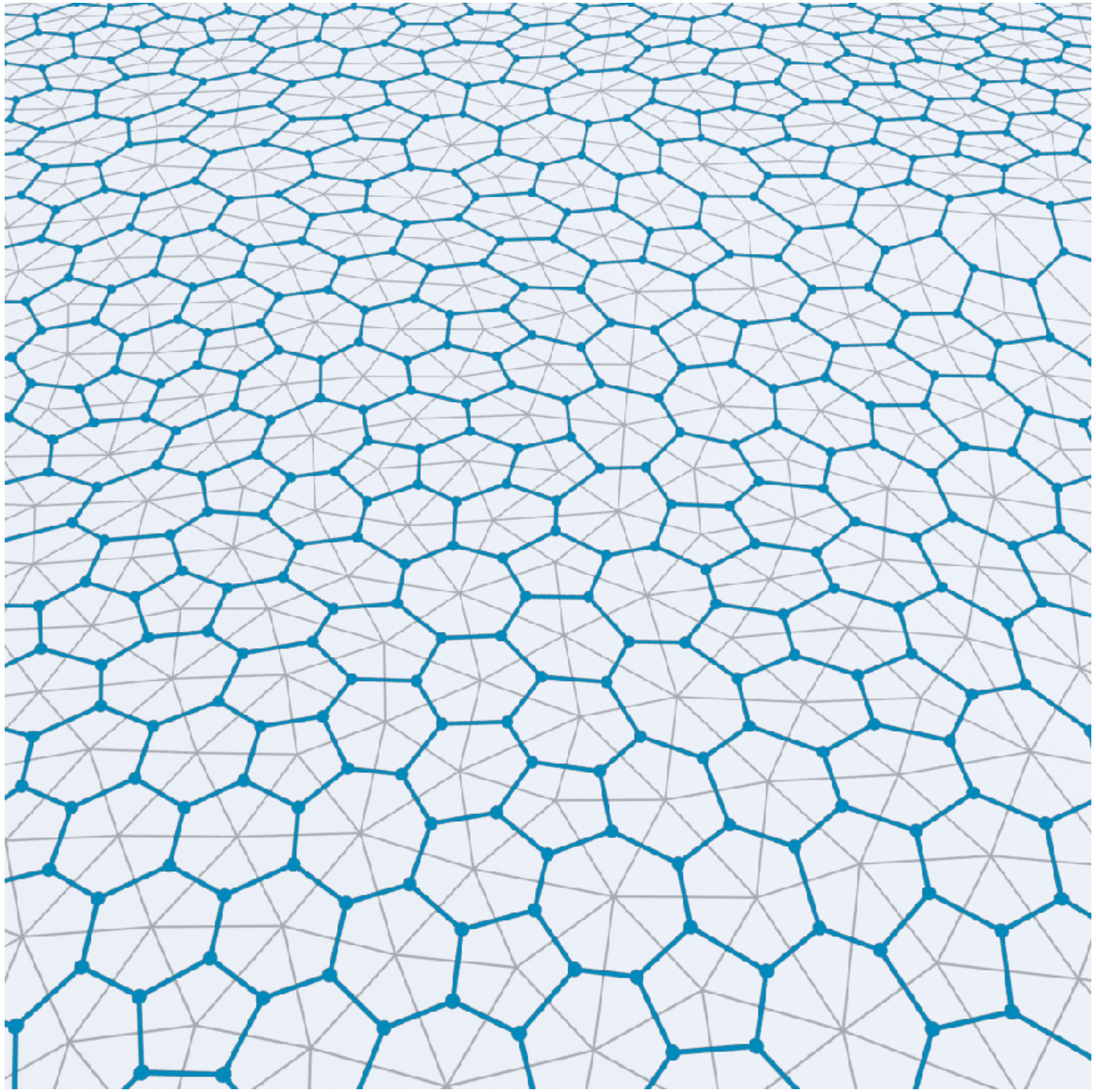

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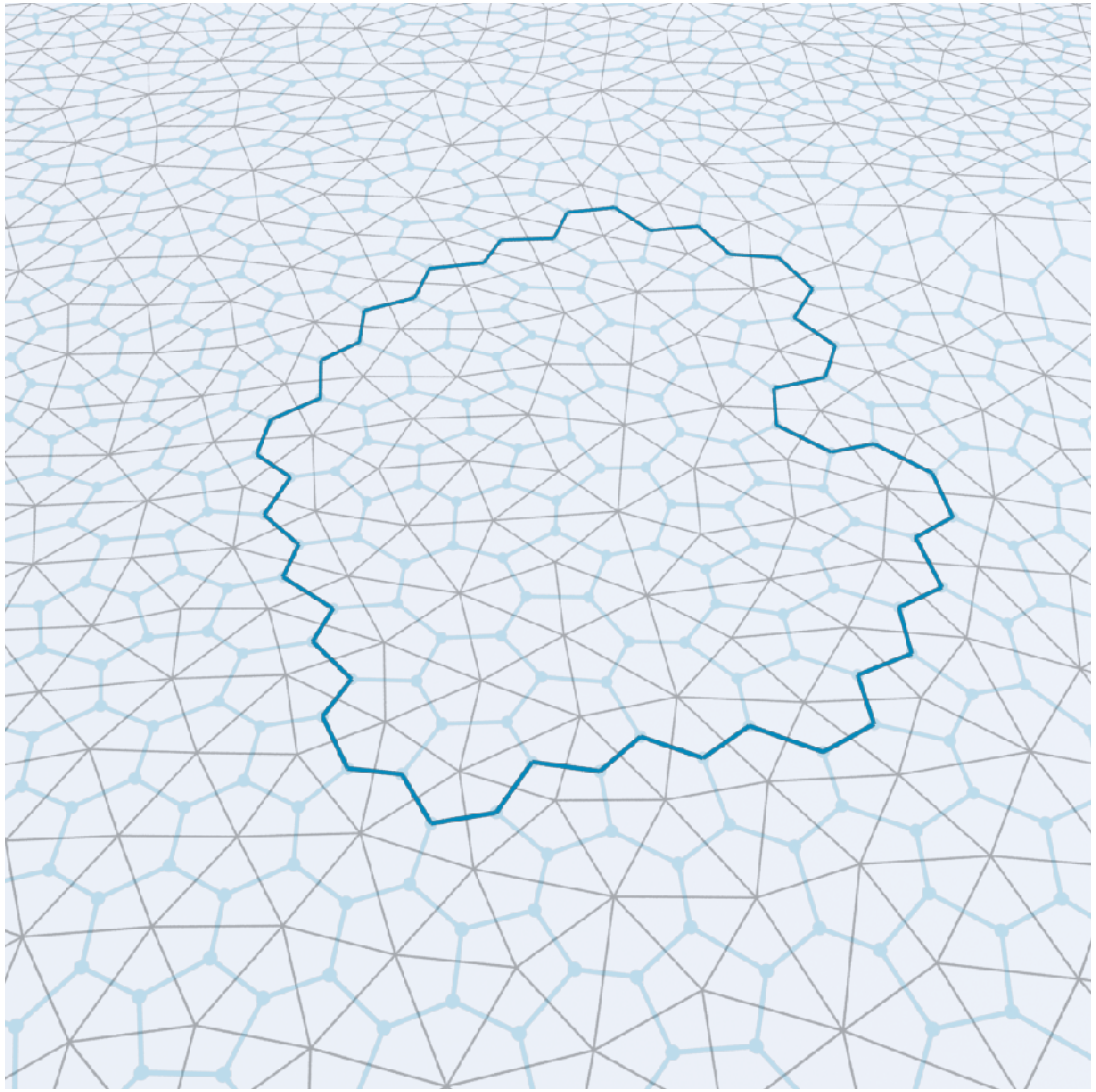

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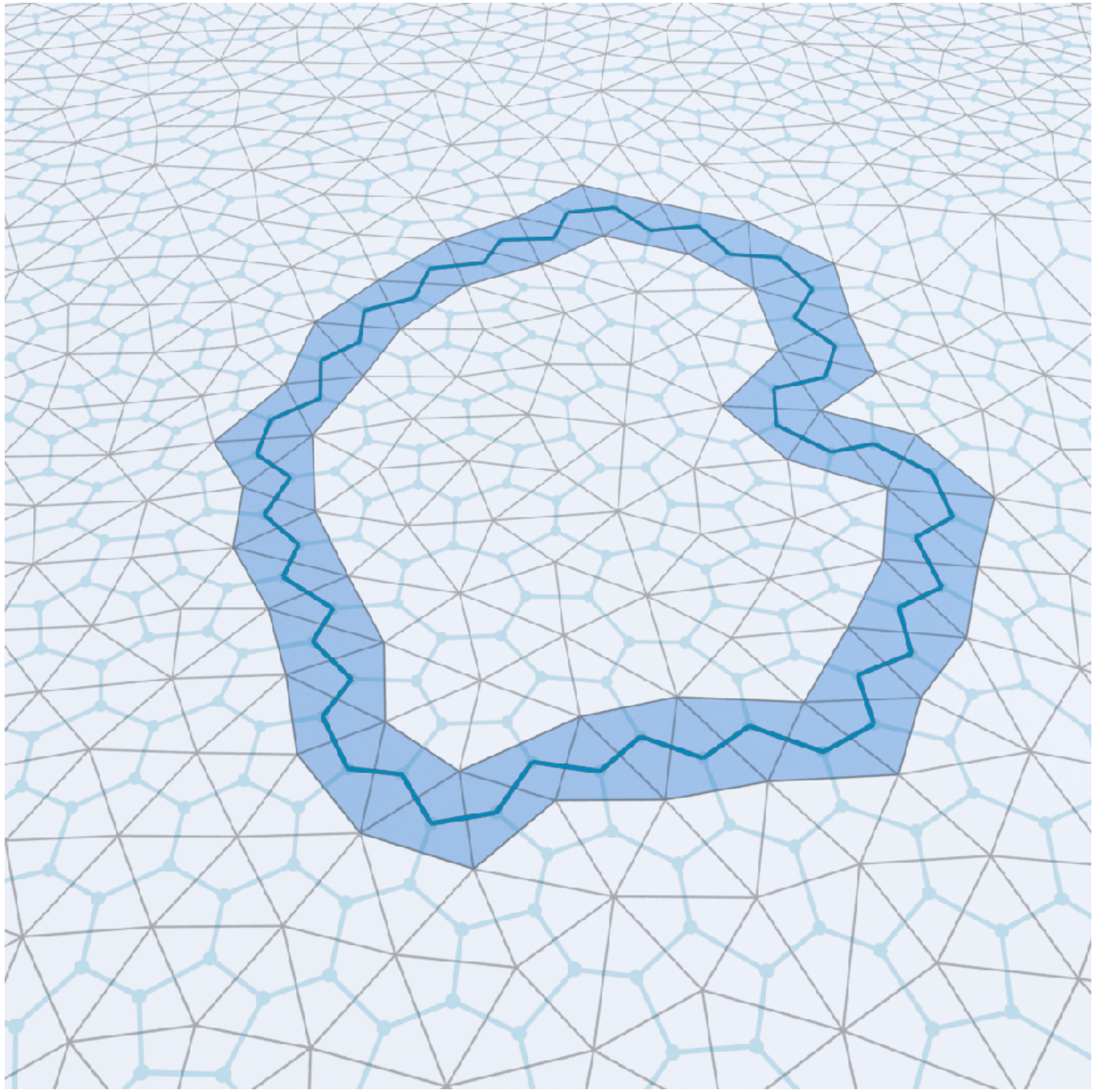
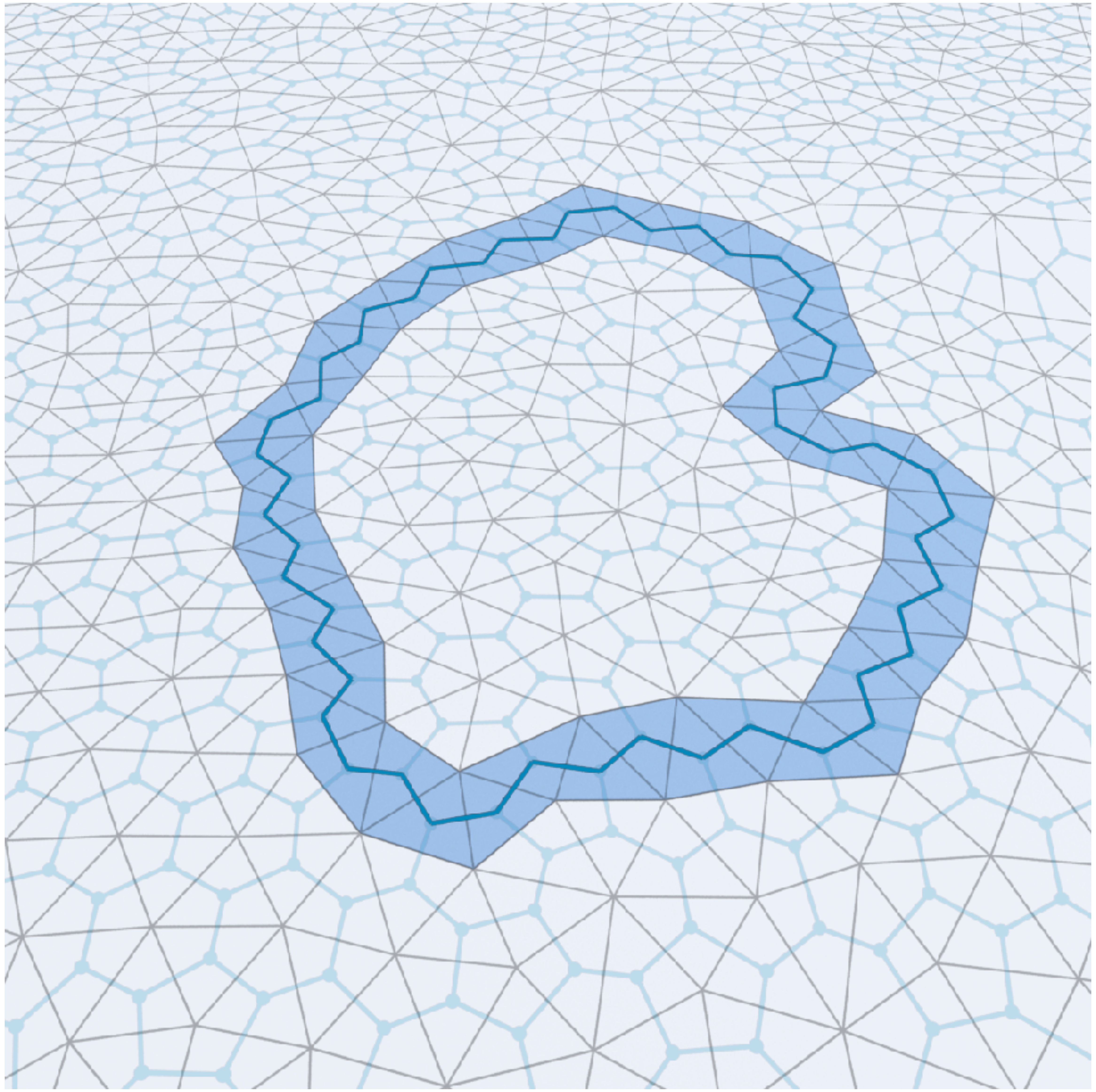


The space of closed strips

The space of closed strips

$$\{\text{closed strips}\}$$

is a vector space over $\mathbb{Z}_2$.



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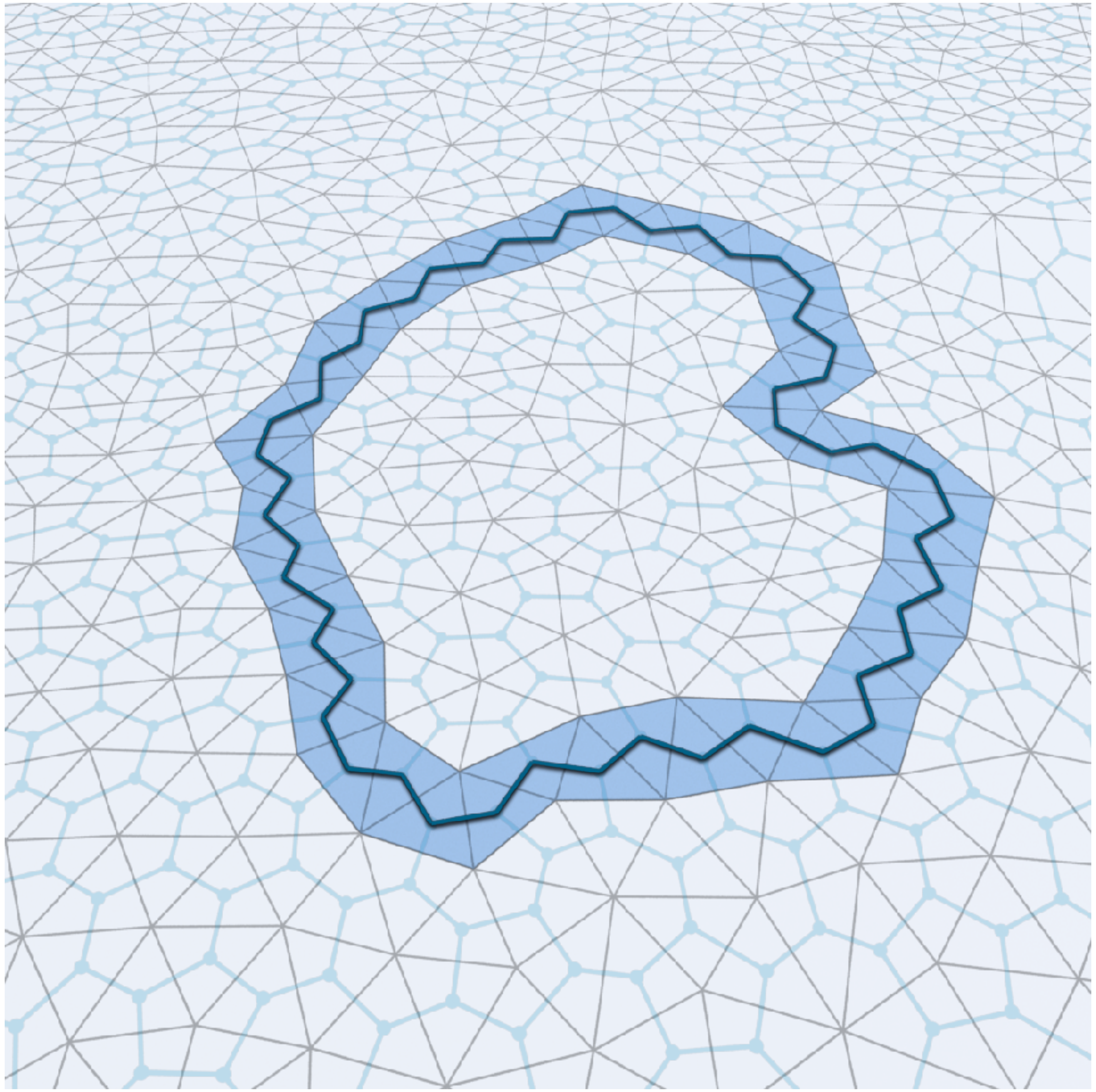

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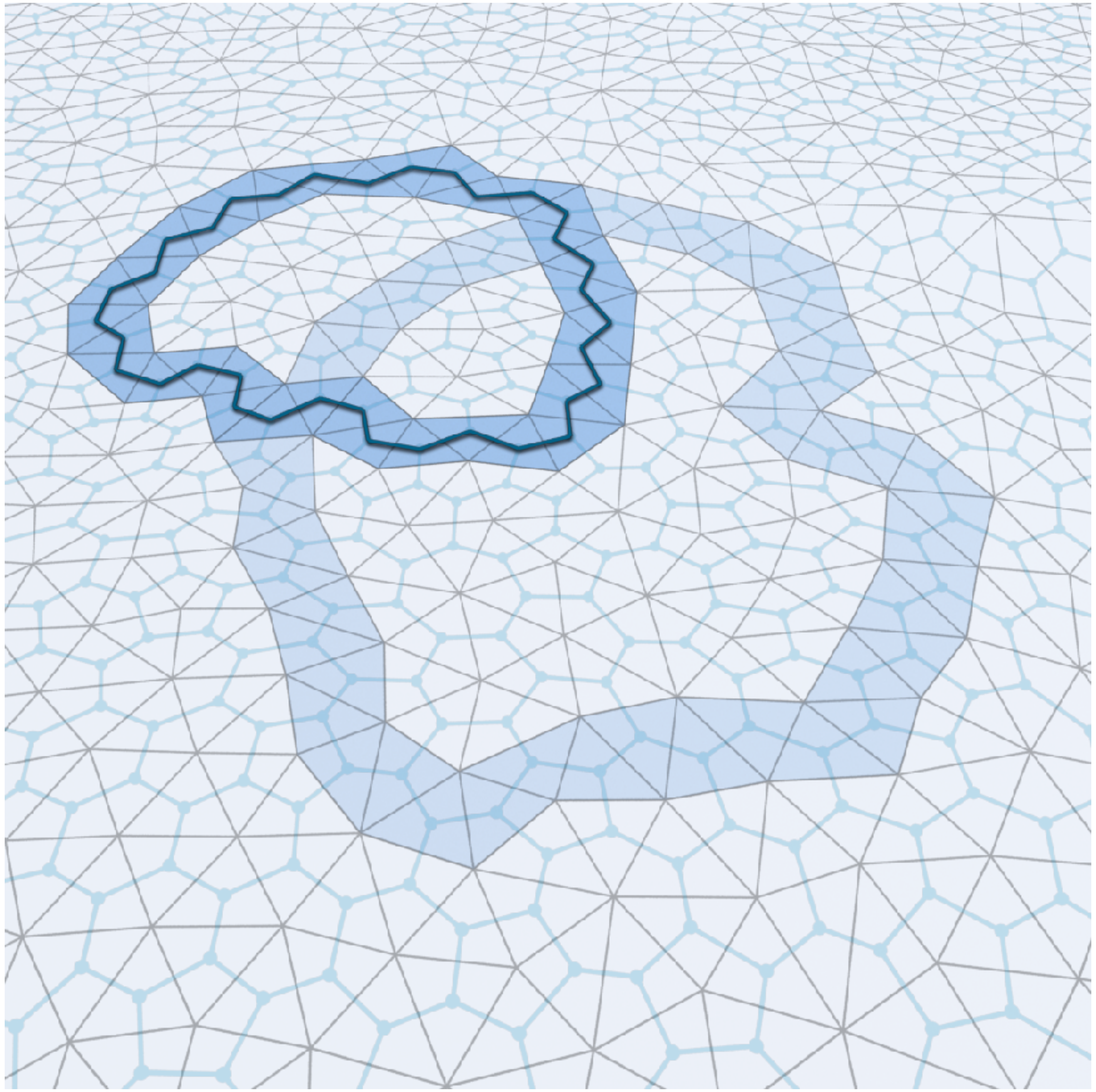

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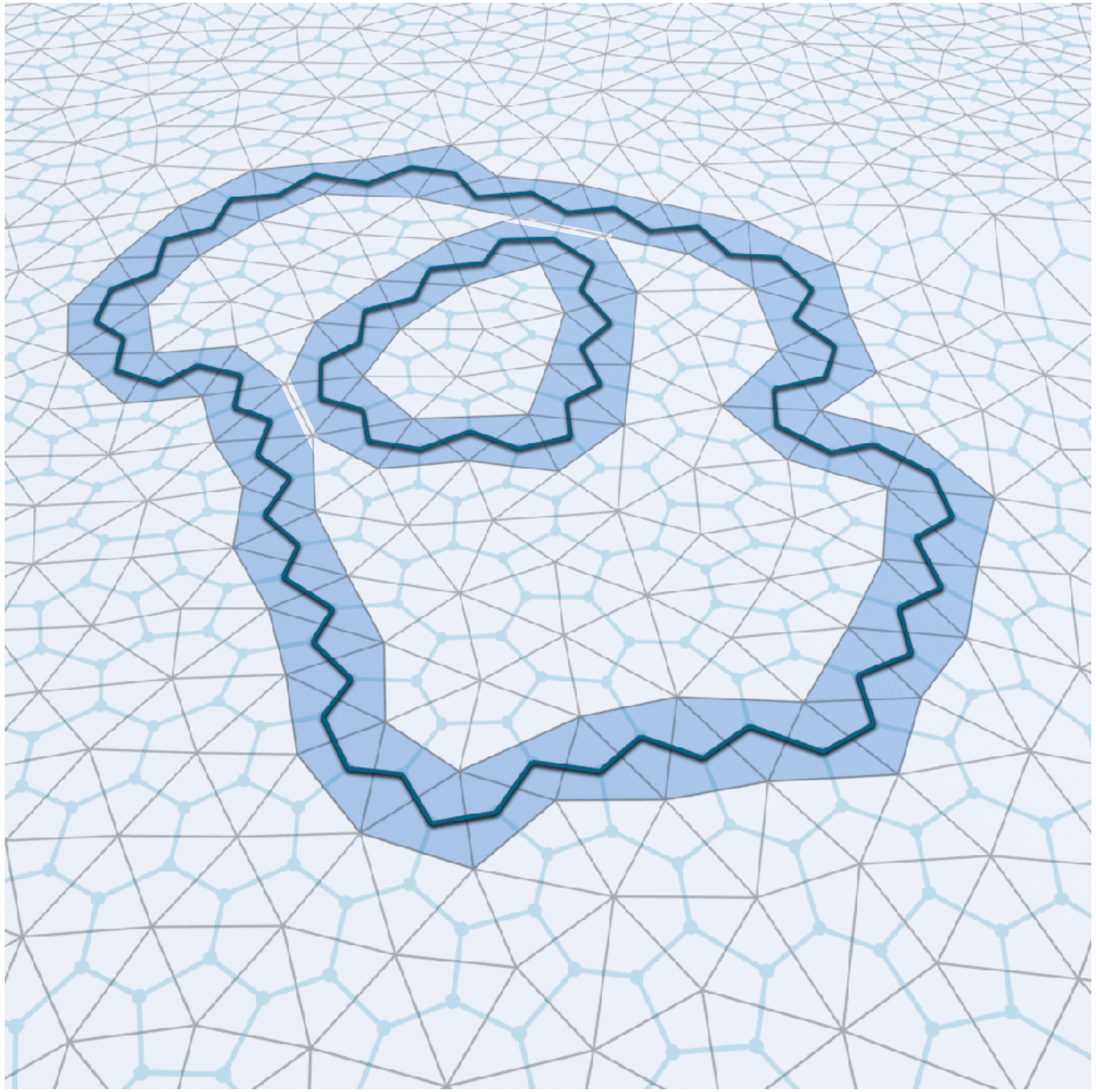
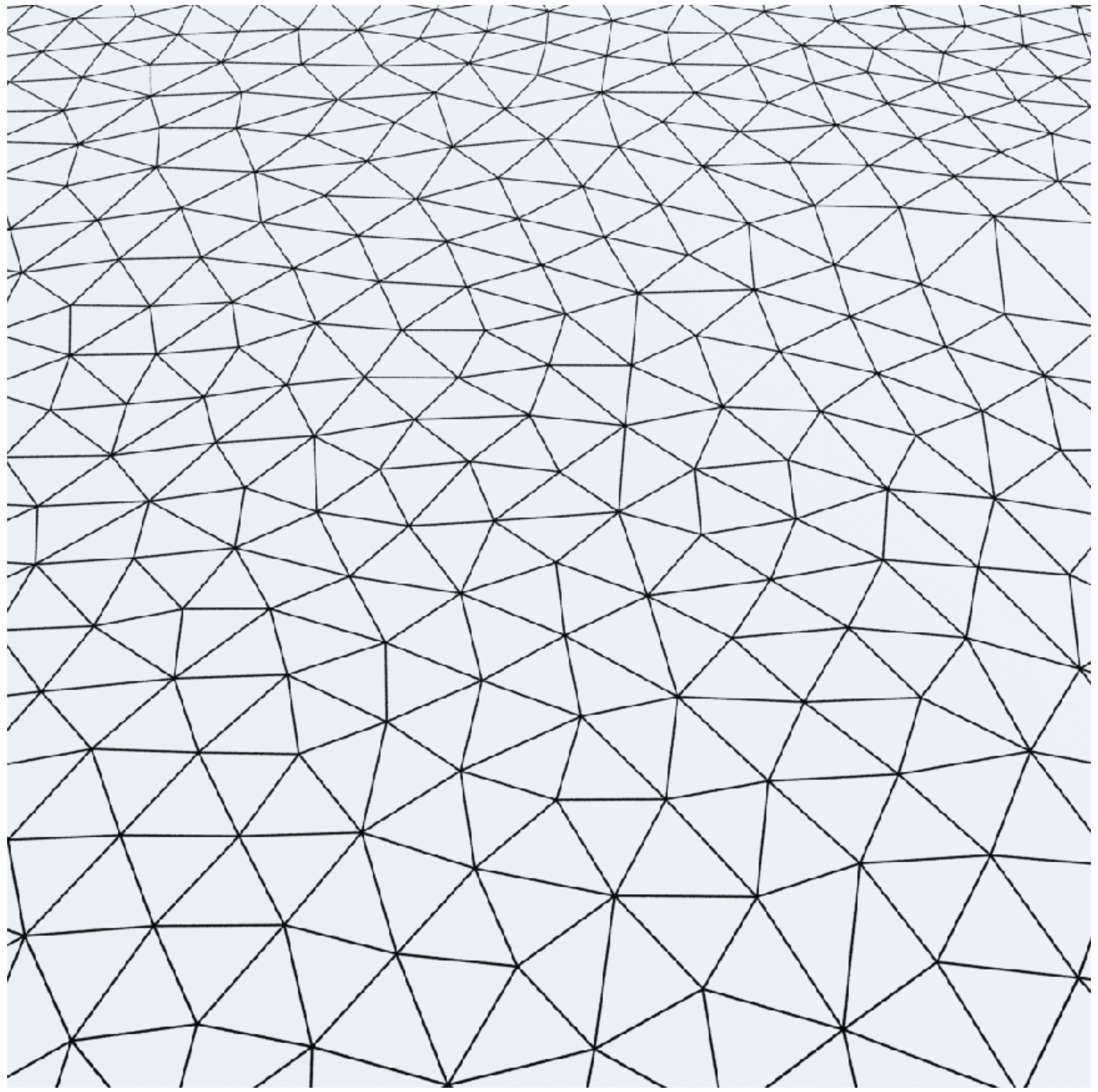
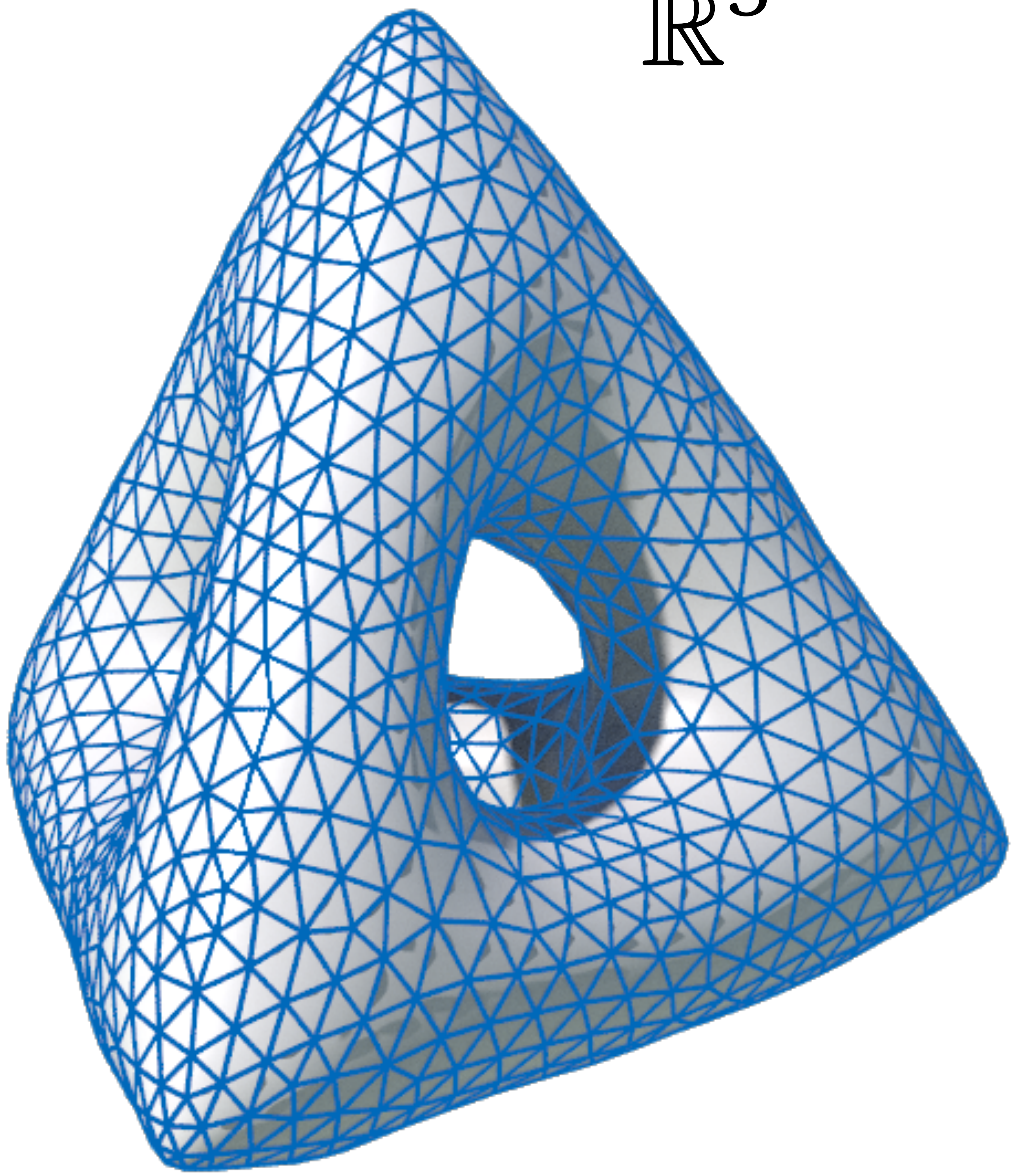


Figure-8/0 function

M



$f : M \to \mathbb{R}^3$



$\mathbb{R}^3$

Figure-8/0 function

$$f : M \to \mathbb{R}^3$$

$$\mathfrak{q}_f : \{\text{closed strips}\} \to \mathbb{Z}_2$$

$$\mathfrak{q}_f(\gamma) = \begin{cases} 0 & \text{if } \gamma \text{ is realized as a Figure-0} \\ 1 & \text{if } \gamma \text{ is realized as a Figure-8} \end{cases}$$

Figure-8/0 function

$\mathfrak{q}_f(\gamma_1) = 0$

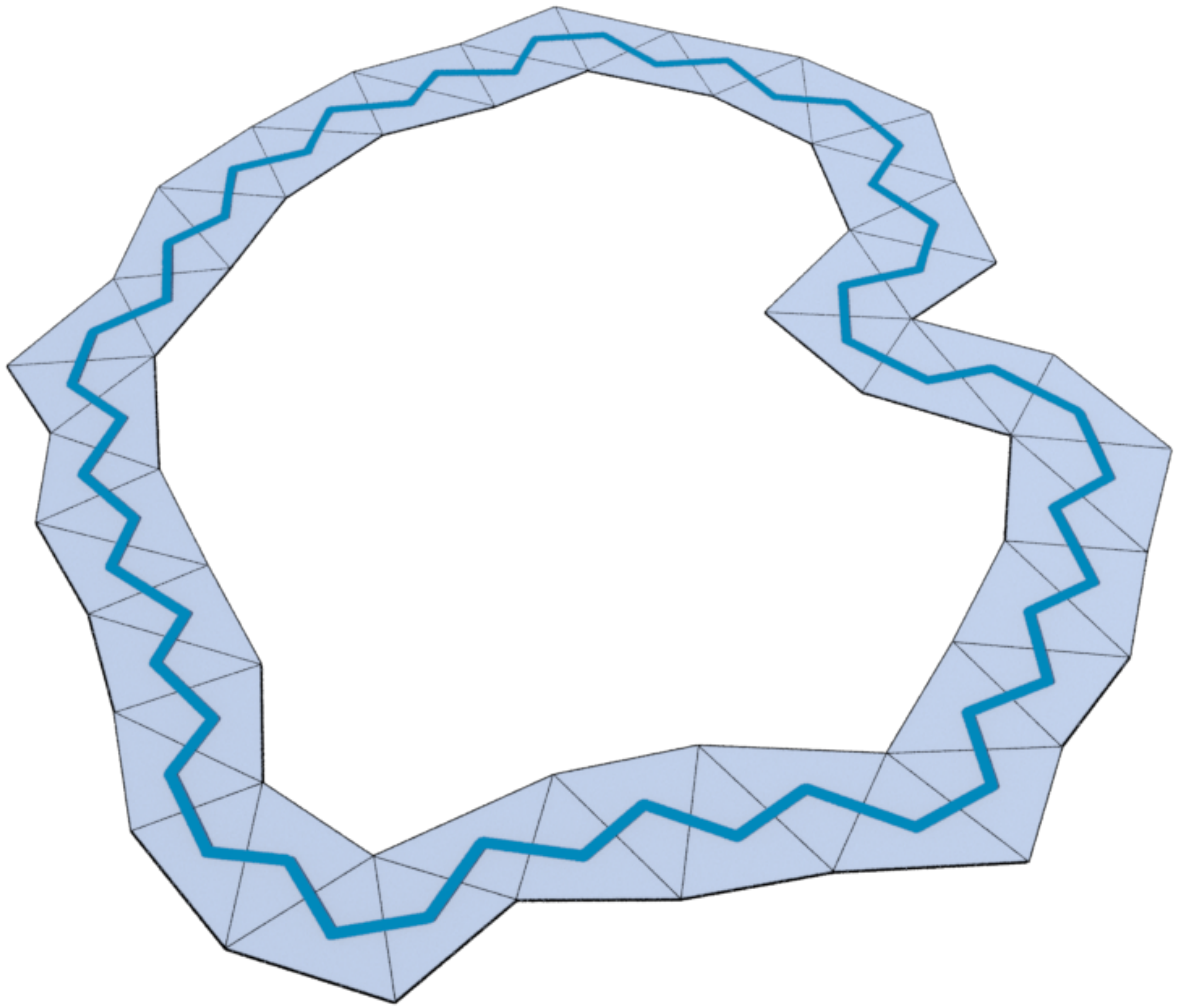


Figure-8/0 function

$$\mathfrak{q}_f(\gamma_1) = 0$$

$$\mathfrak{q}_f(\gamma_2) = 0$$

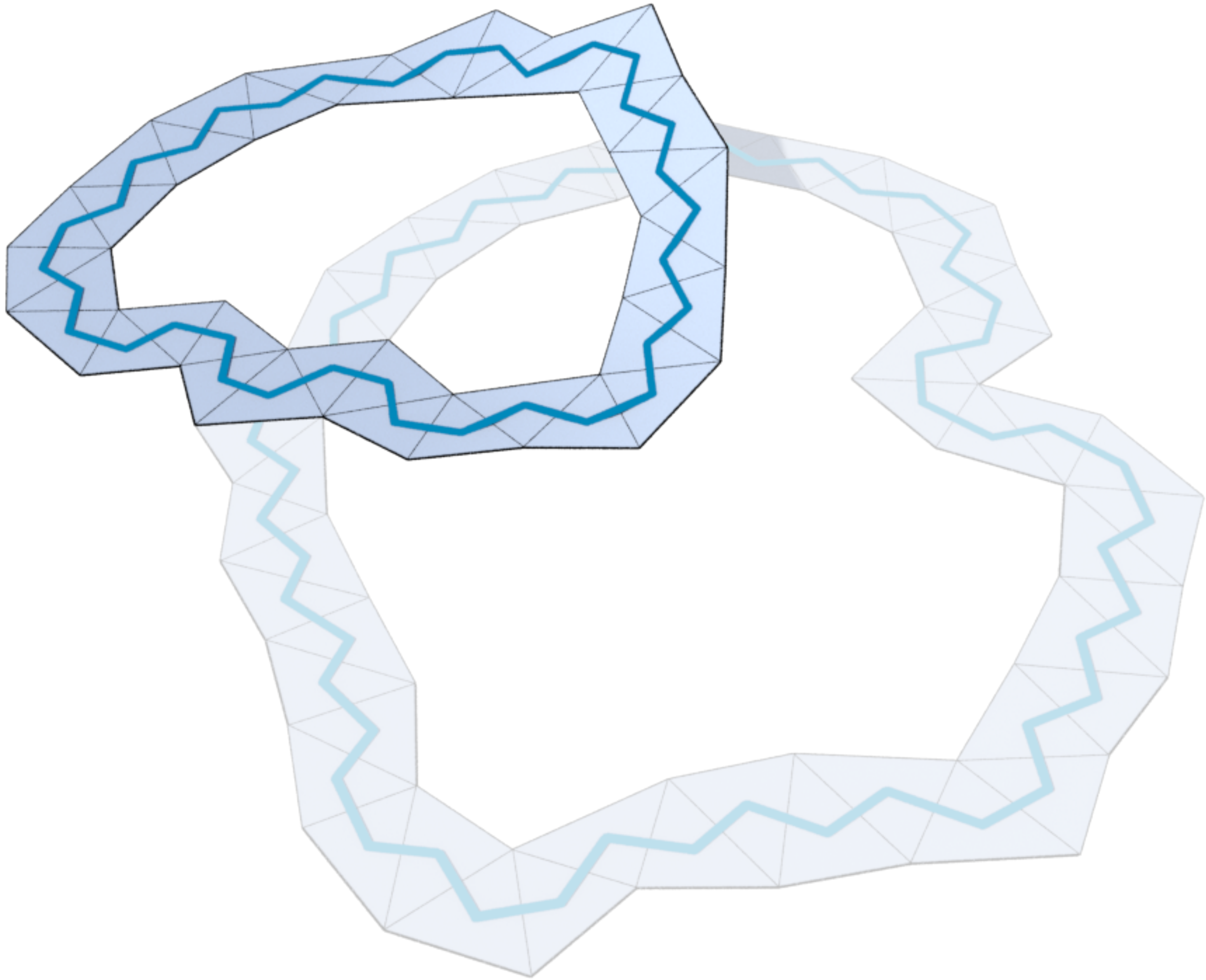


Figure-8/0 function

$$\mathfrak{q}_f(\gamma_1) = 0$$
$$\mathfrak{q}_f(\gamma_2) = 0$$
$$[\gamma_1 \cap \gamma_2] = 1$$

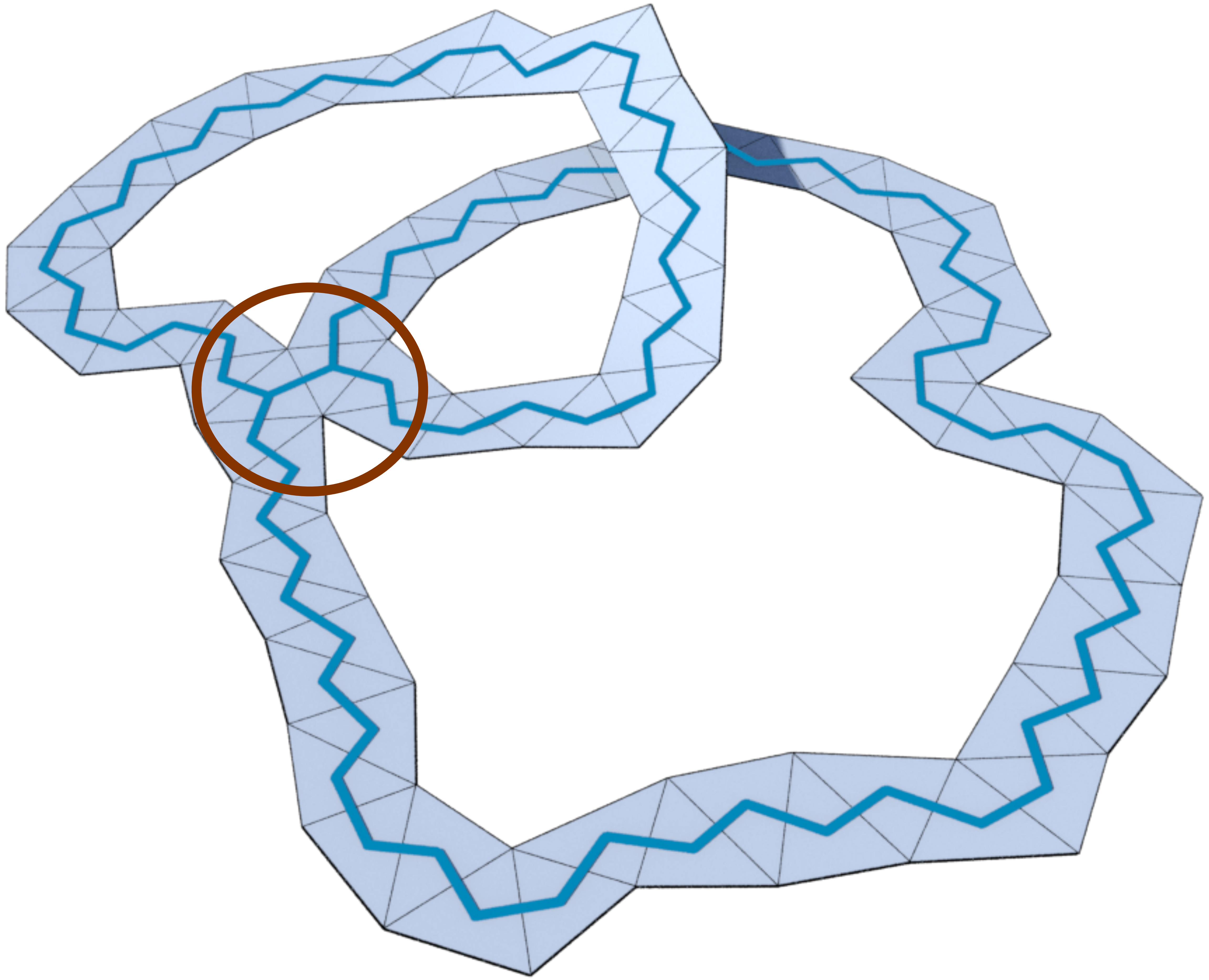


Figure-8/0 function

$$\mathfrak{q}_f(\gamma_1) = 0$$
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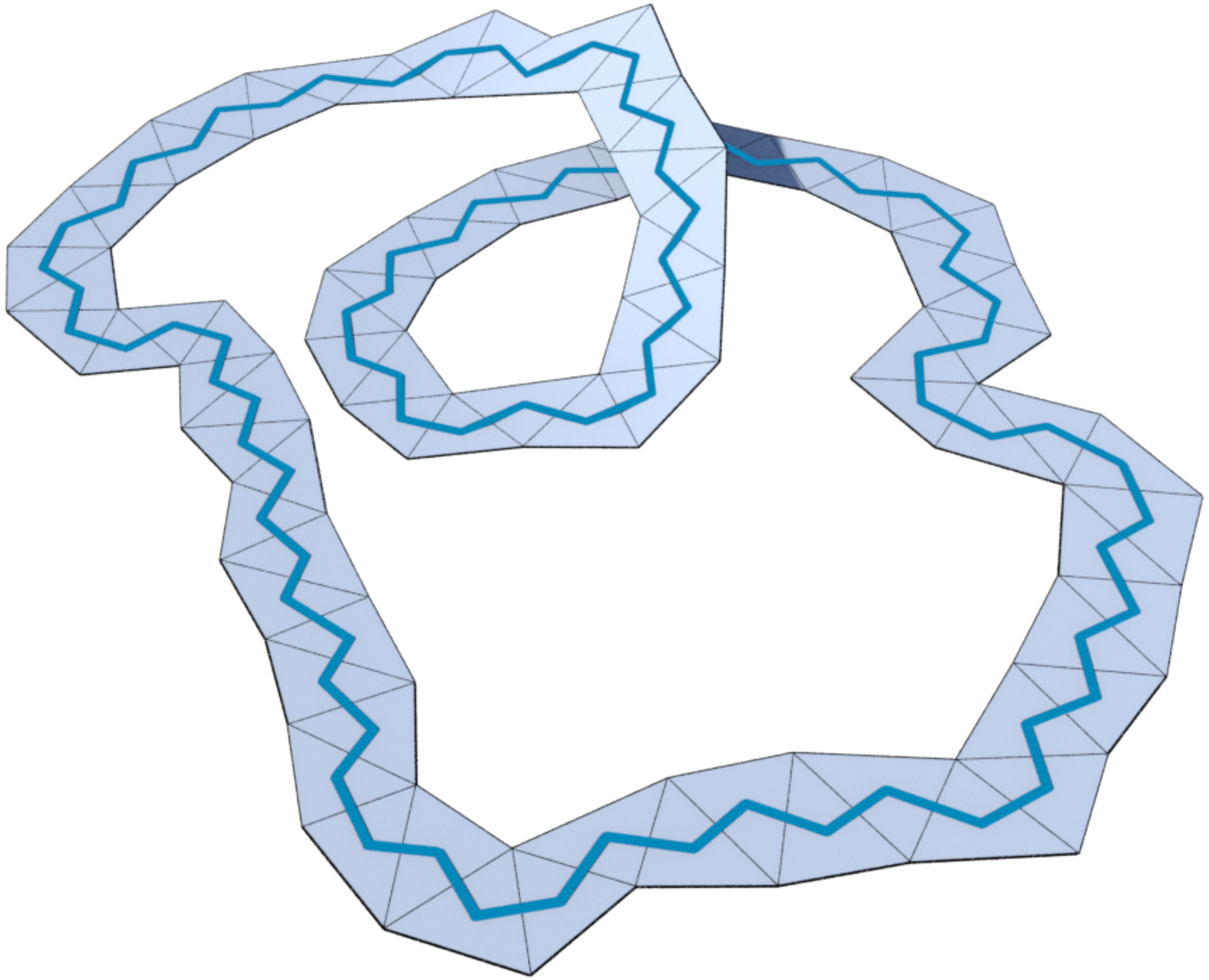


Figure-8/0 function

$$\mathfrak{q}_f(\gamma_1) = 0$$
$$\mathfrak{q}_f(\gamma_2) = 0$$
$$[\gamma_1 \cap \gamma_2] = 1$$
$$\mathfrak{q}_f(\gamma_1 + \gamma_2) = 1$$

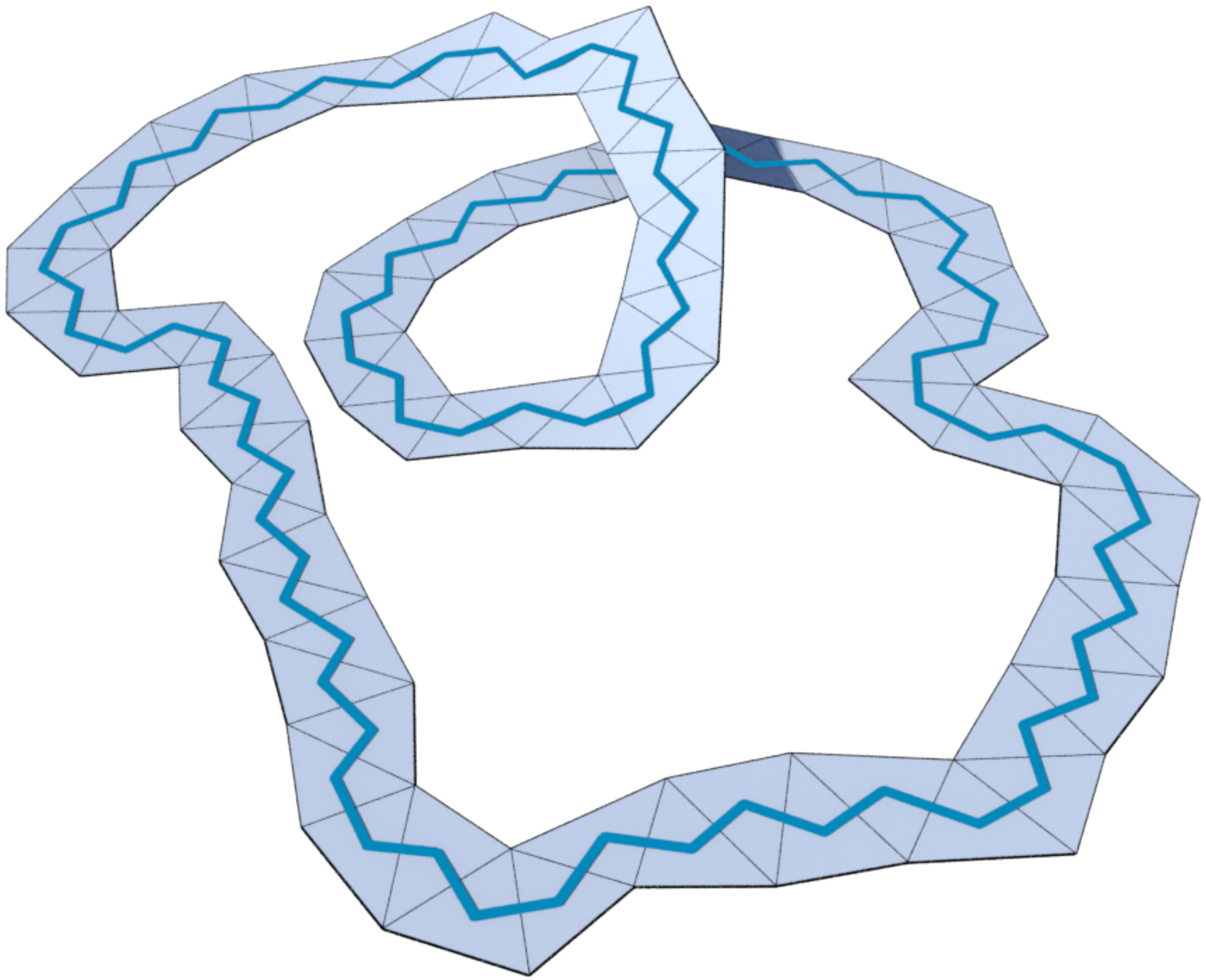
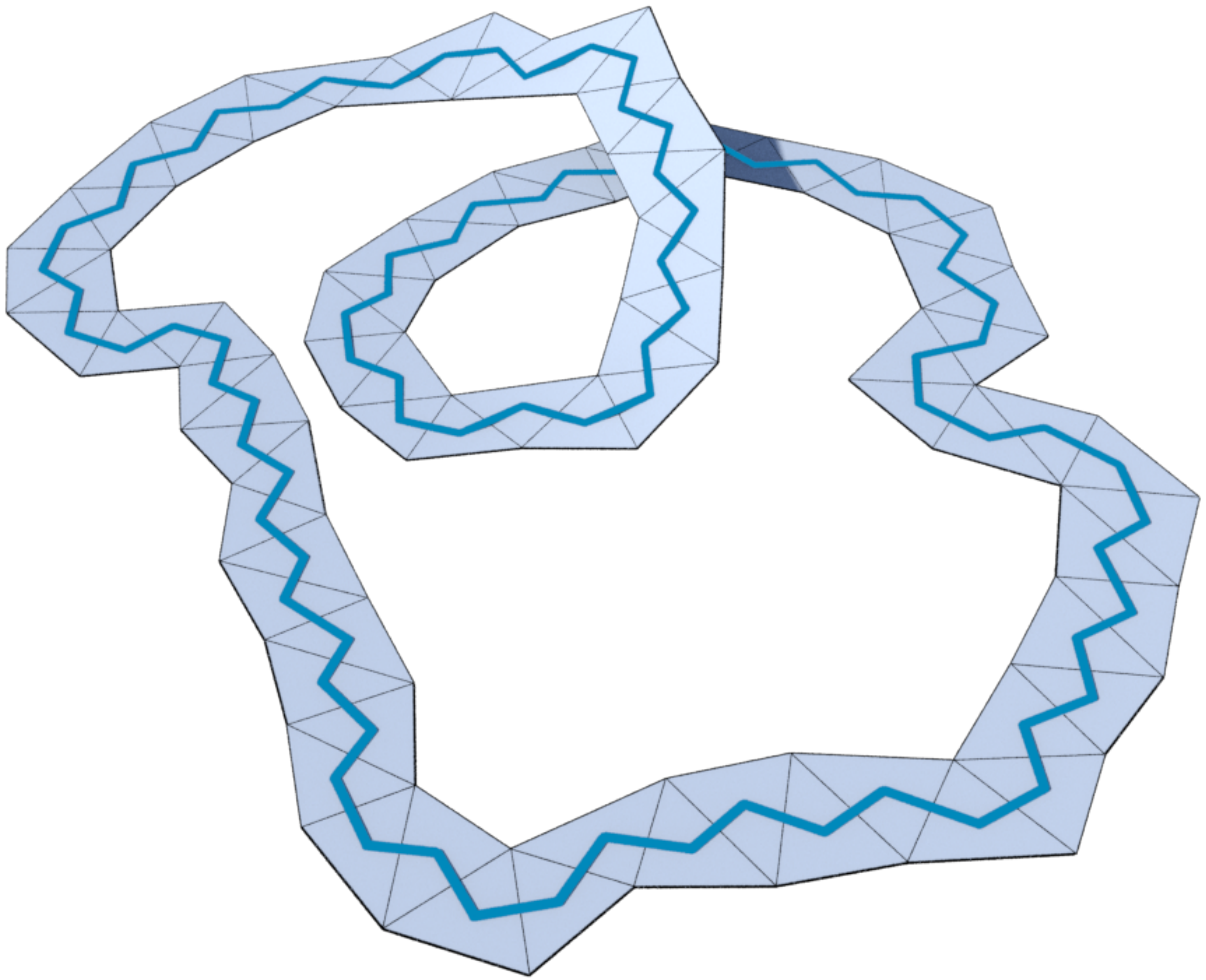


Figure-8/0 function

$$\mathfrak{q}_f(\gamma_1 + \gamma_2) = \mathfrak{q}_f(\gamma_1) + \mathfrak{q}_f(\gamma_2) + [\gamma_1 \cap \gamma_2]$$

$$\mathfrak{q}_f(\gamma_1) = 0$$
$$\mathfrak{q}_f(\gamma_2) = 0$$
$$[\gamma_1 \cap \gamma_2] = 1$$
$$\mathfrak{q}_f(\gamma_1 + \gamma_2) = 1$$



Quadratic forms

$$\mathfrak{q}_f(\gamma_1 + \gamma_2) = \mathfrak{q}_f(\gamma_1) + \mathfrak{q}_f(\gamma_2) + [\gamma_1 \cap \gamma_2]$$

$\mathfrak{q}_f$ is a **quadratic form** associated with the scalar product $[\cdot \cap \cdot]$ on the $\mathbb{Z}_2$ vector space $\{\text{closed strips}\}$.

There are *many* quadratic forms associated with the same scalar product when the space is over a finite field of characteristic 2.

Quadratic forms

Suppose $\mathfrak{q}, \tilde{\mathfrak{q}}$ are two quadratic forms associated with $[\cdot \cap \cdot]$,

$$\mathfrak{q}(\gamma_1 + \gamma_2) = \mathfrak{q}(\gamma_1) + \mathfrak{q}(\gamma_2) + [\gamma_1 \cap \gamma_2]$$

$$\tilde{\mathfrak{q}}(\gamma_1 + \gamma_2) = \tilde{\mathfrak{q}}(\gamma_1) + \tilde{\mathfrak{q}}(\gamma_2) + [\gamma_1 \cap \gamma_2]$$

Quadratic forms

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$$-)\quad \tilde{\mathfrak{q}}(\gamma_1 + \gamma_2) = \tilde{\mathfrak{q}}(\gamma_1) + \tilde{\mathfrak{q}}(\gamma_2) + [\gamma_1 \cap \gamma_2]$$

$$(\mathfrak{q} - \tilde{\mathfrak{q}})(\gamma_1 + \gamma_2) = (\mathfrak{q} - \tilde{\mathfrak{q}})(\gamma_1) + (\mathfrak{q} - \tilde{\mathfrak{q}})(\gamma_2)$$

The ***difference*** of two such quadratic forms is a ***linear functional***.

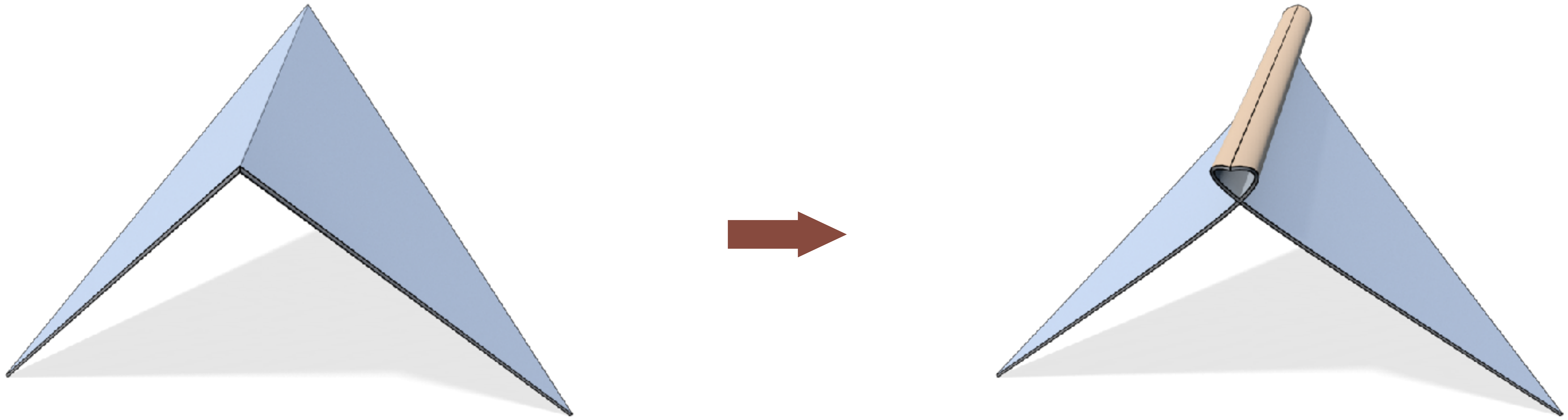
Quadratic forms

The ***difference*** of two such quadratic forms is a ***linear functional***.

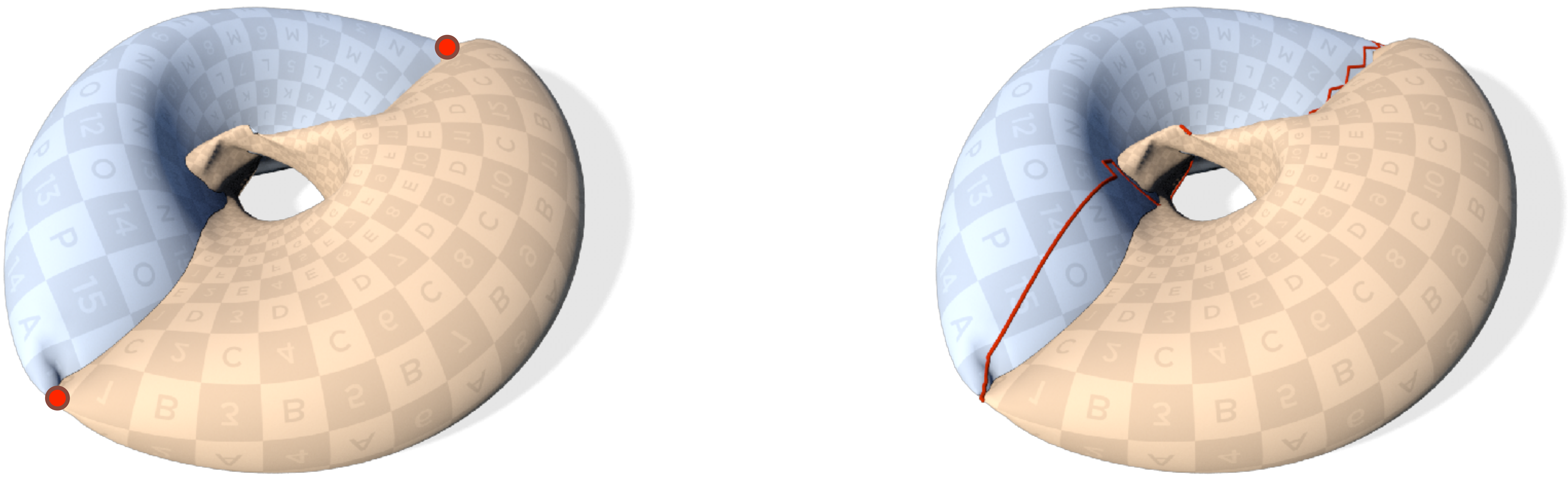
The collection of these quadratic forms is an ***affine space*** parallel to $\{\text{closed strips}\}^*$.

The geometric representations of elements in $\{\text{closed strips}\}^*$ are ***rims***.

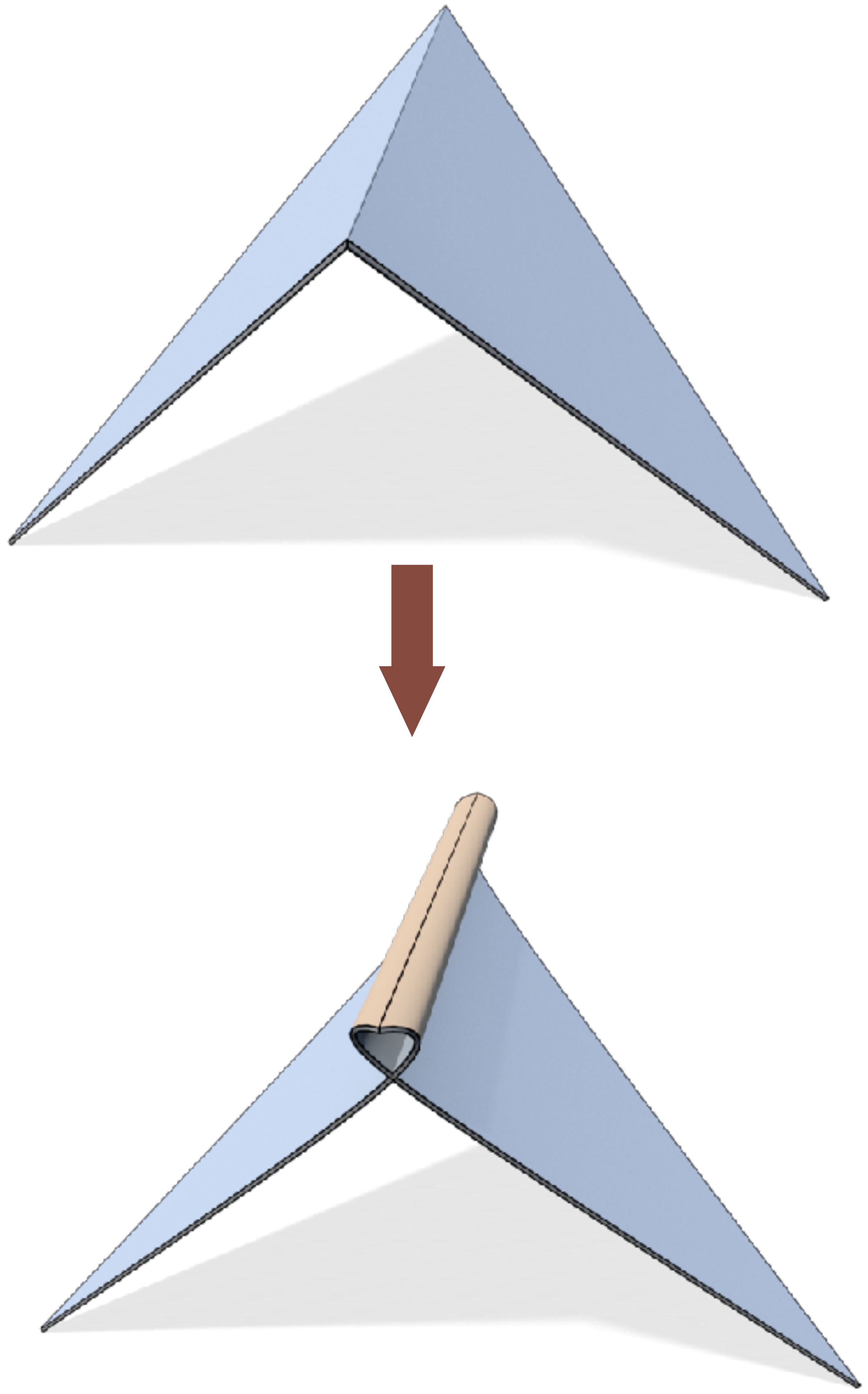
Rims

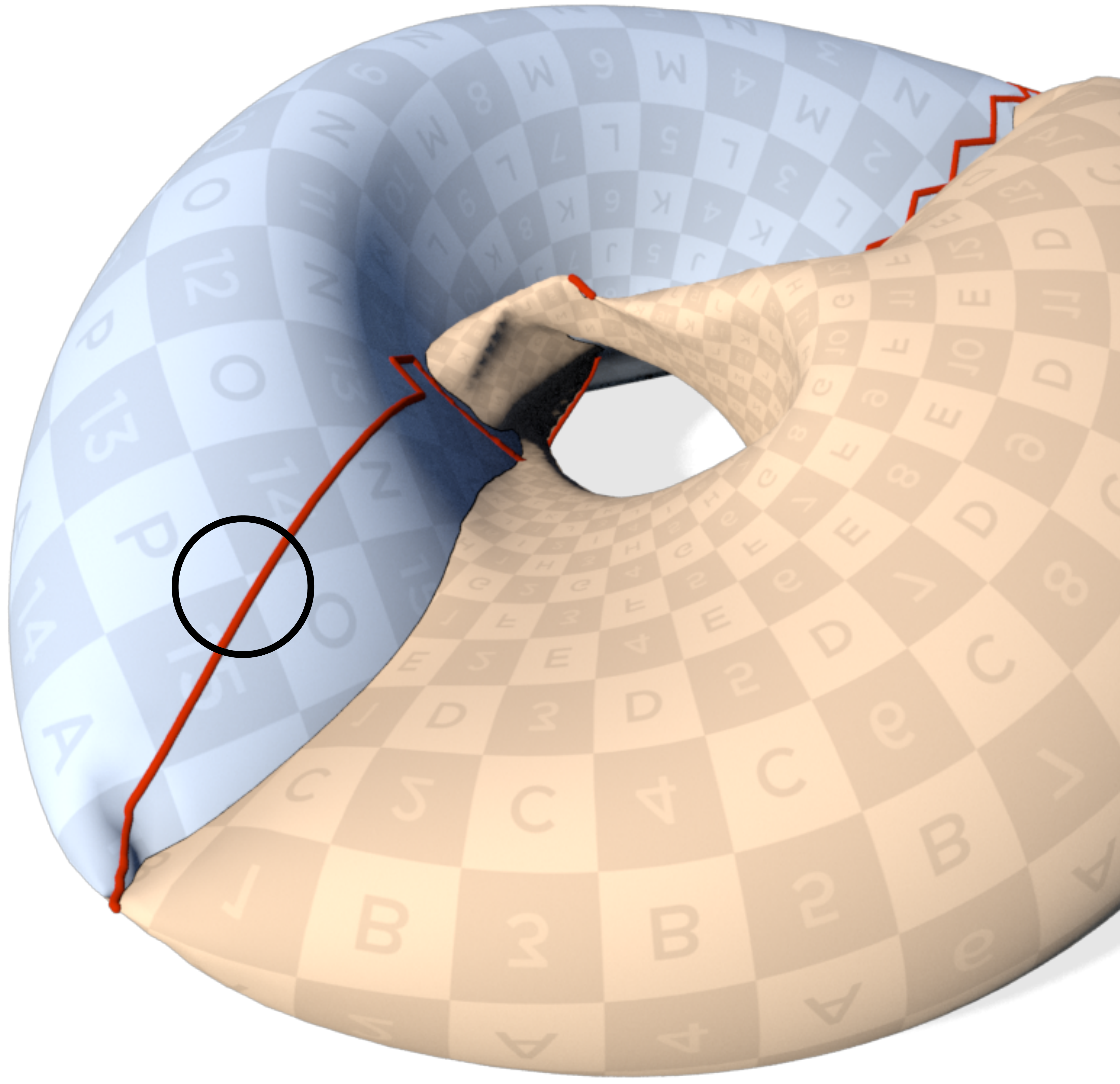


Rims

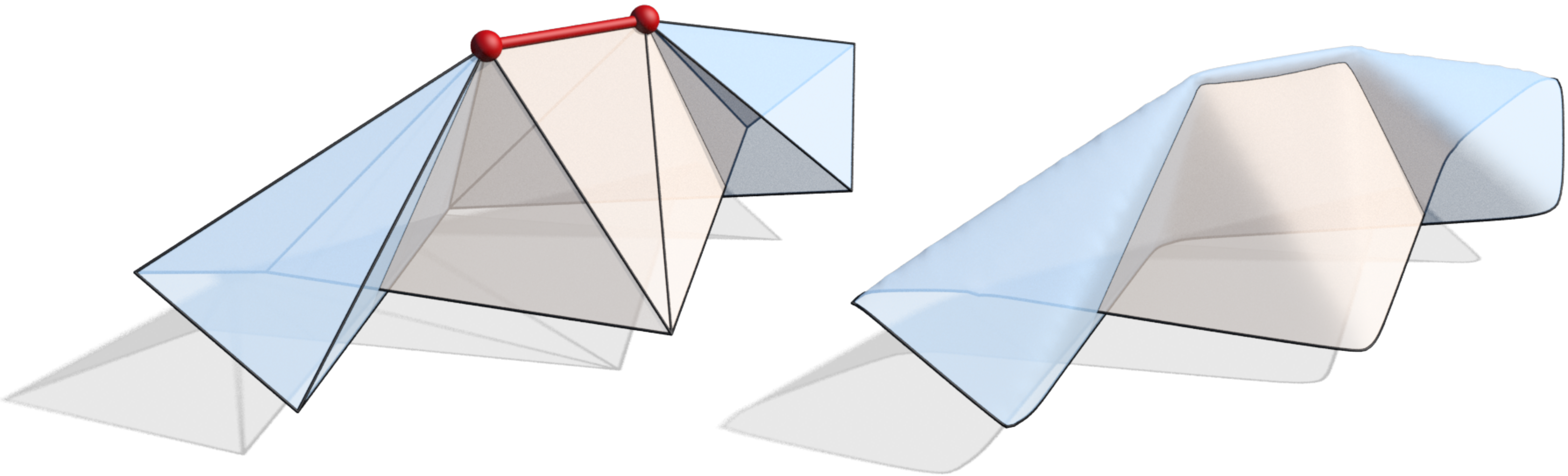


Rims





Rims



Rimmed surface

A **rimmed surface** $(f, \mathfrak{s})$ consists of

- a surface realization $f : M \to \mathbb{R}^3$
- rims $\mathfrak{s} \in C_1(M, \partial M; \mathbb{Z}_2) \cong C_1(M^*; \mathbb{Z}_2)^*$

The Figure-8/0 function for a rimmed surface $(f, \mathfrak{s})$ is given by

$$\mathfrak{q}_{(f,\mathfrak{s})} = \mathfrak{q}_f + \mathfrak{s}$$

Rimmed surface

- The Figure-8/0 type of strips is described algebraically by a **quadratic form** $\mathfrak{q}$.

- With a prescribed $\mathfrak{q}$, any surface realization

$$f : M \to \mathbb{R}^3$$

shall be decorated with **rims** $\mathfrak{s} \in \mathfrak{q} - \mathfrak{q}_f$.

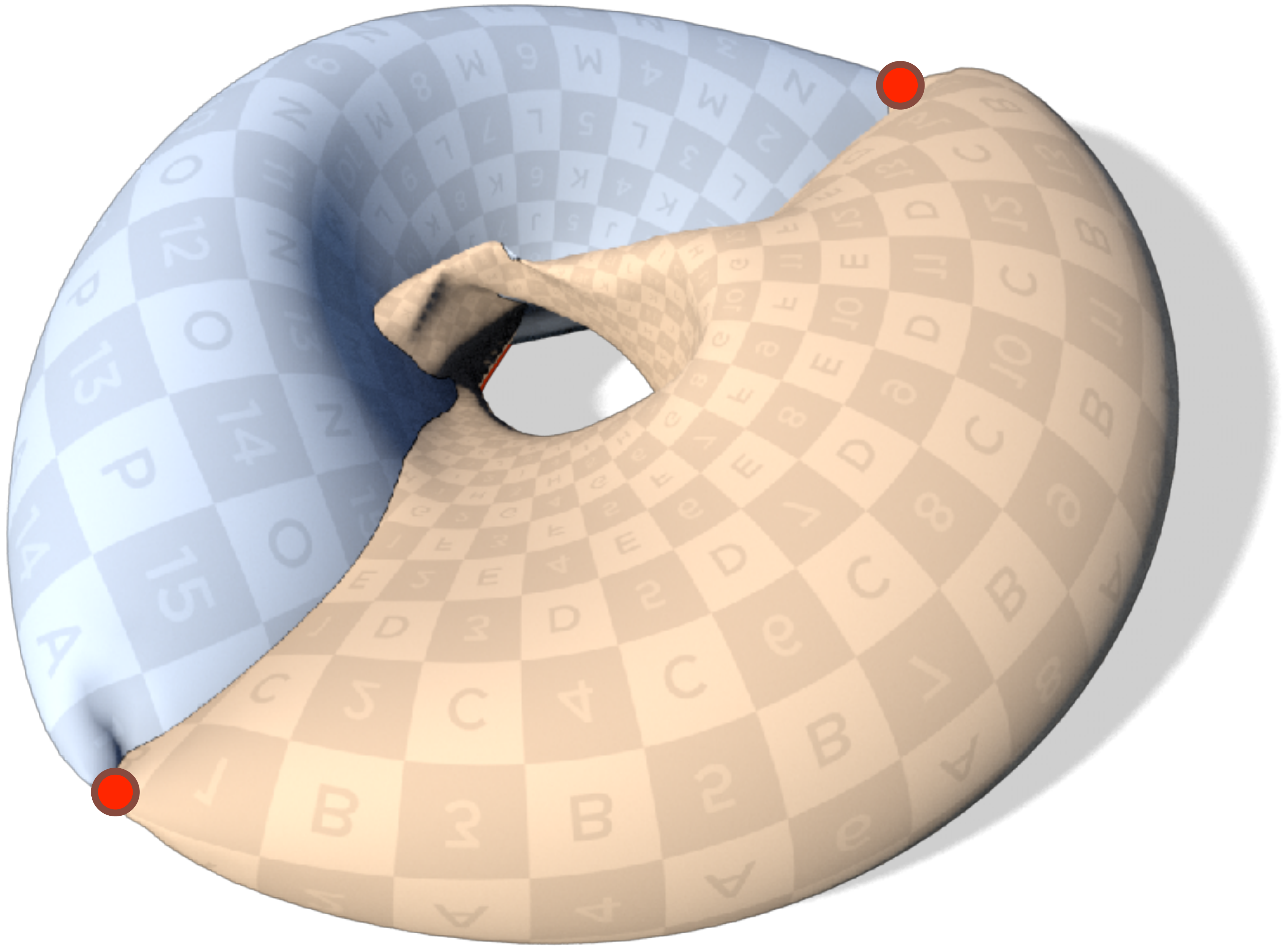
Emergent surface

Microscopic scale

Setting up gauge field r_{ij}

Macroscopic scale

minimize $\displaystyle\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_\epsilon^2$



Emergent surface

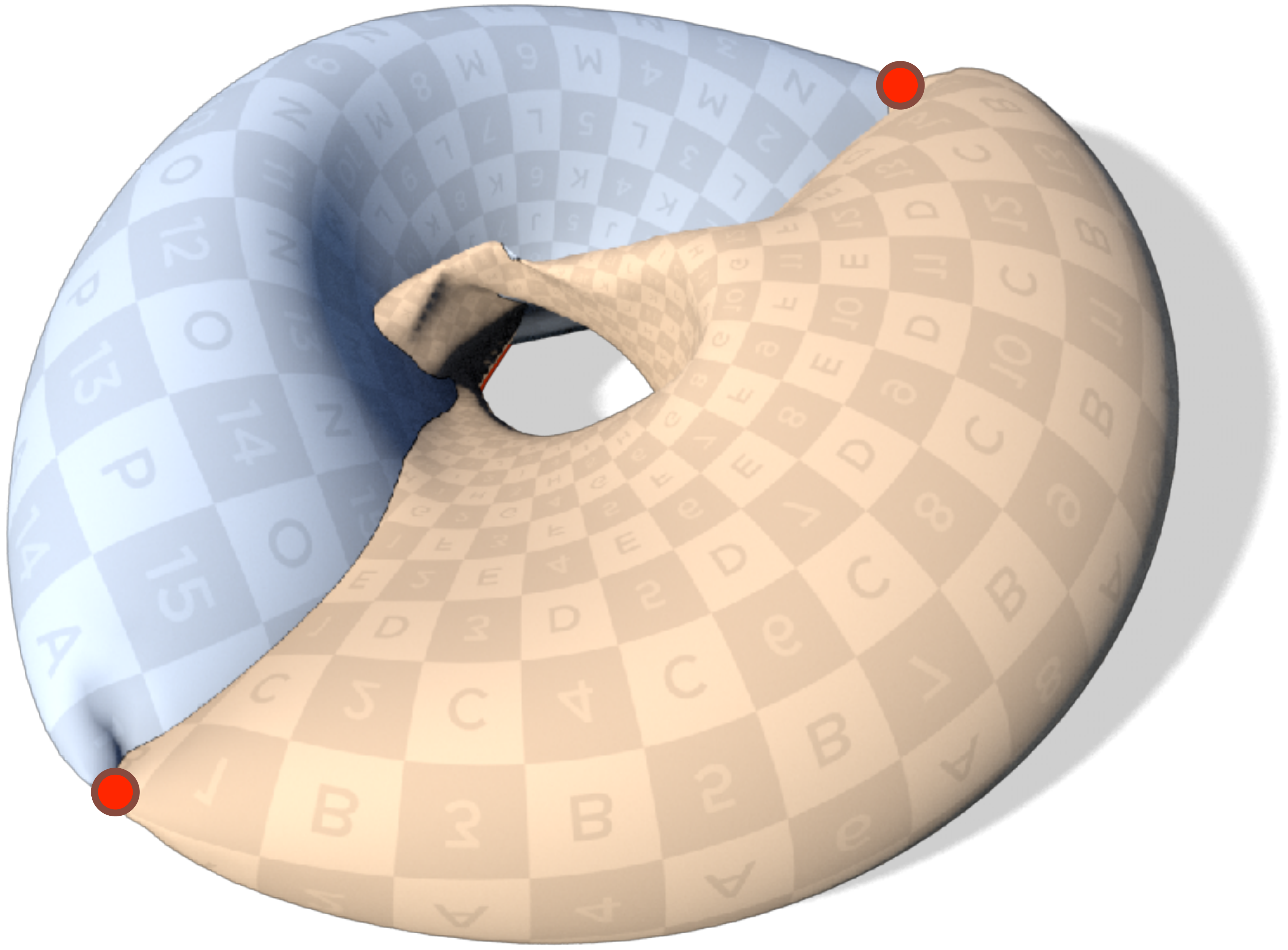
Microscopic scale

Setting up gauge field r_{ij}

and a quadratic form $\mathfrak{q}$

Macroscopic scale

minimize $\displaystyle\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_\epsilon^2$



Emergent surface

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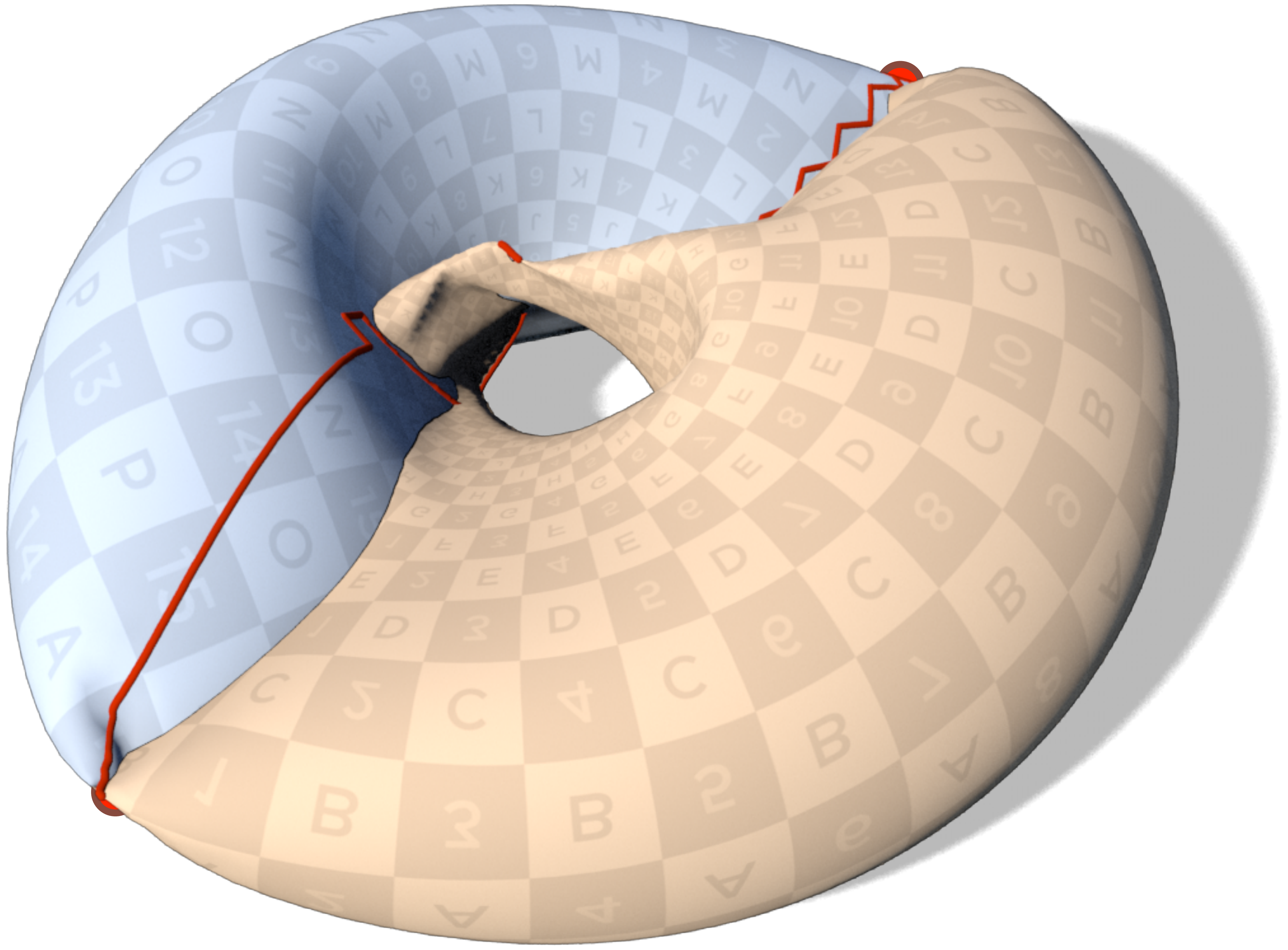
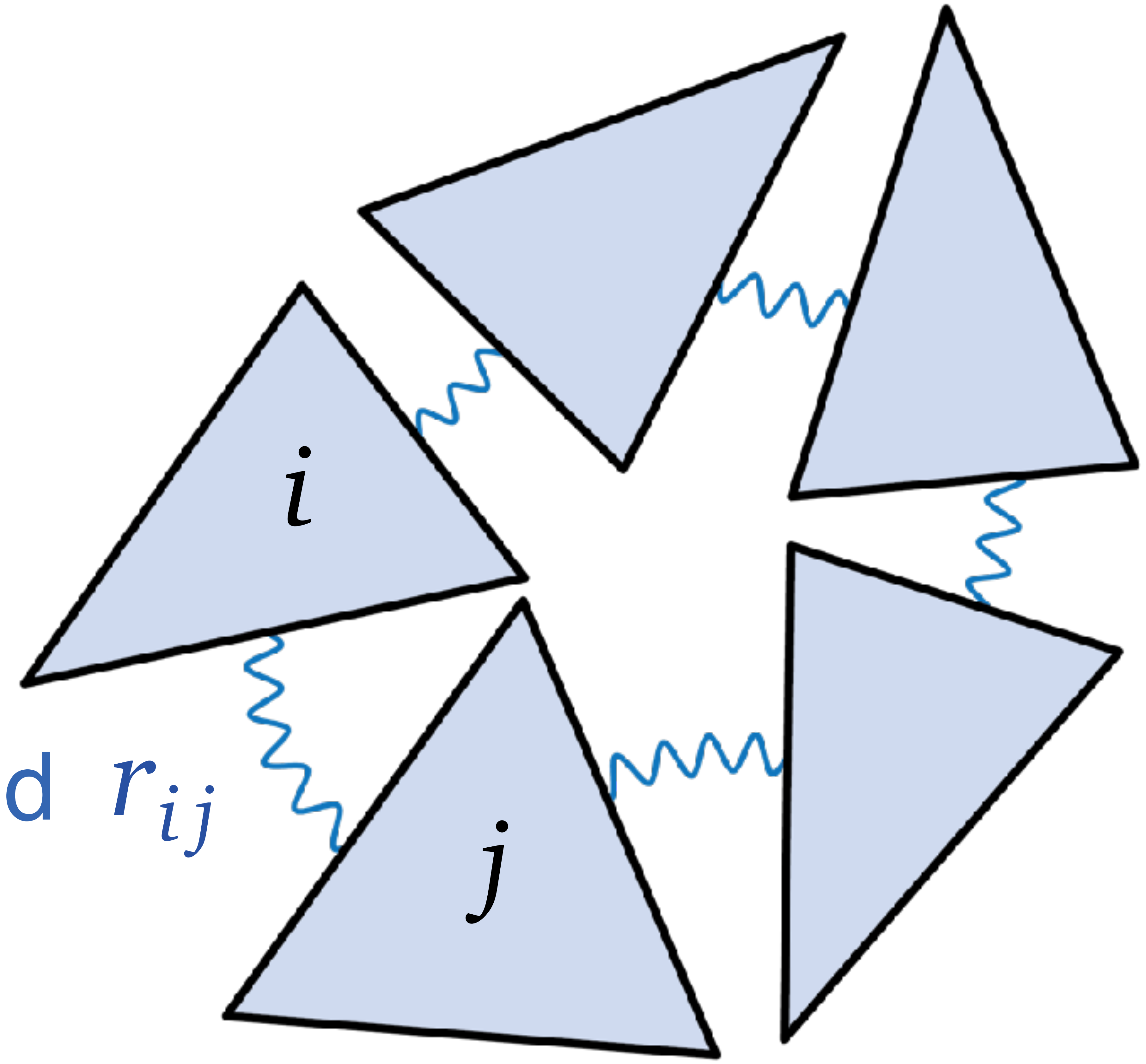
minimize $\displaystyle\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_{\epsilon}^{2}$

minimize $|\mathfrak{s}| = |\mathfrak{q}_f - \mathfrak{q}|$

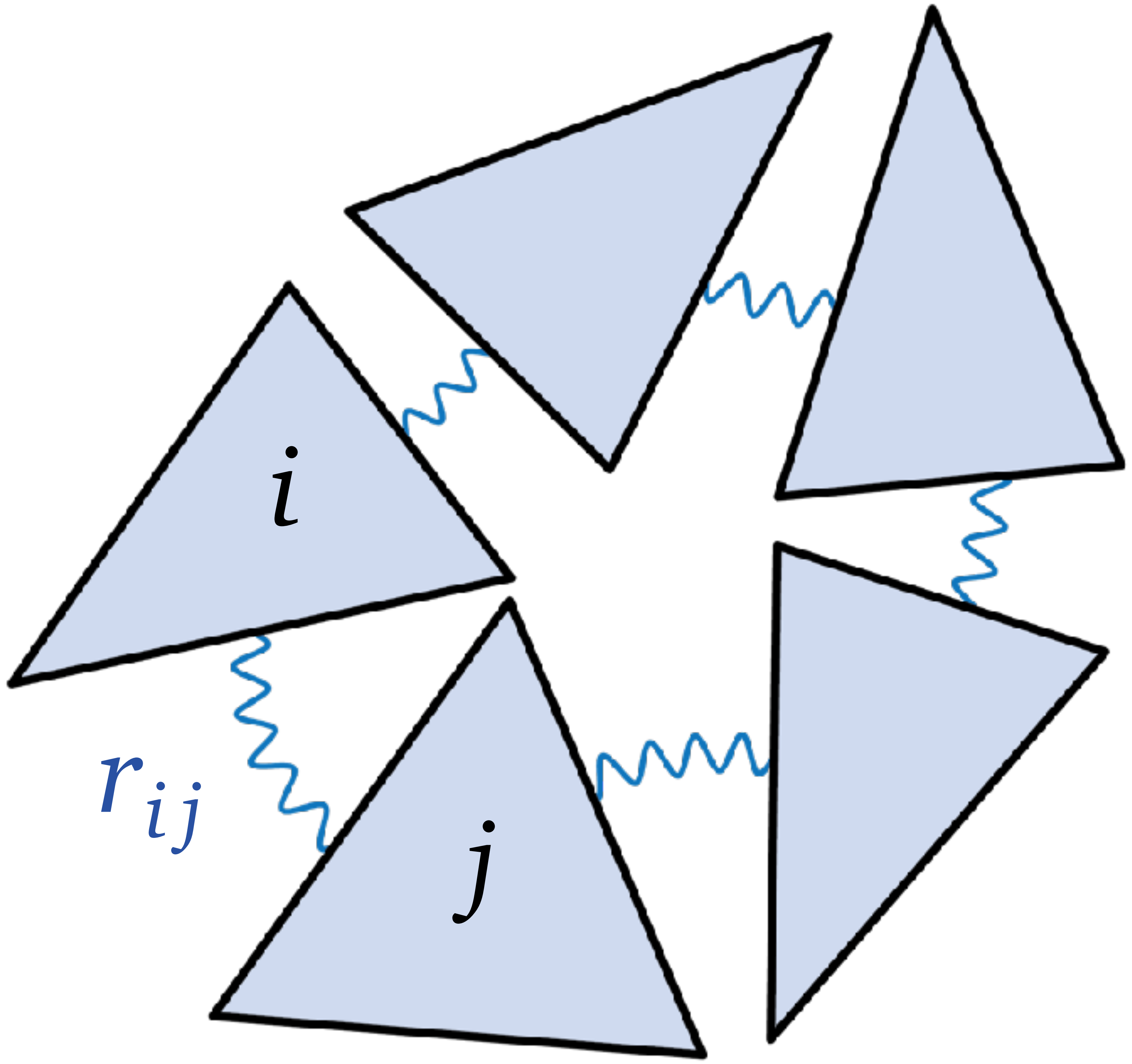


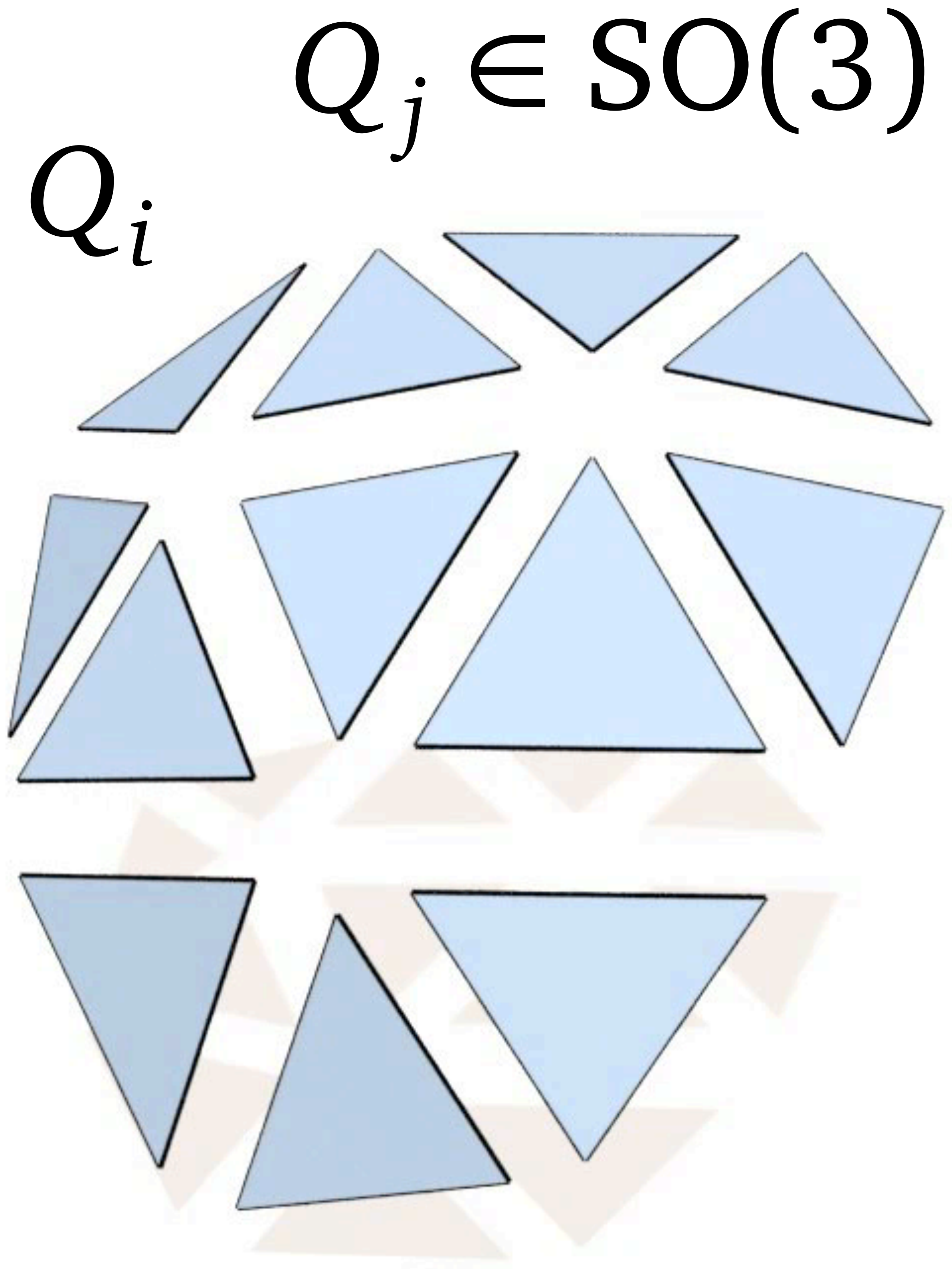
Lifting rotations to spinors

Rotational gauge field r_{ij}



Lifting rotations to spinors





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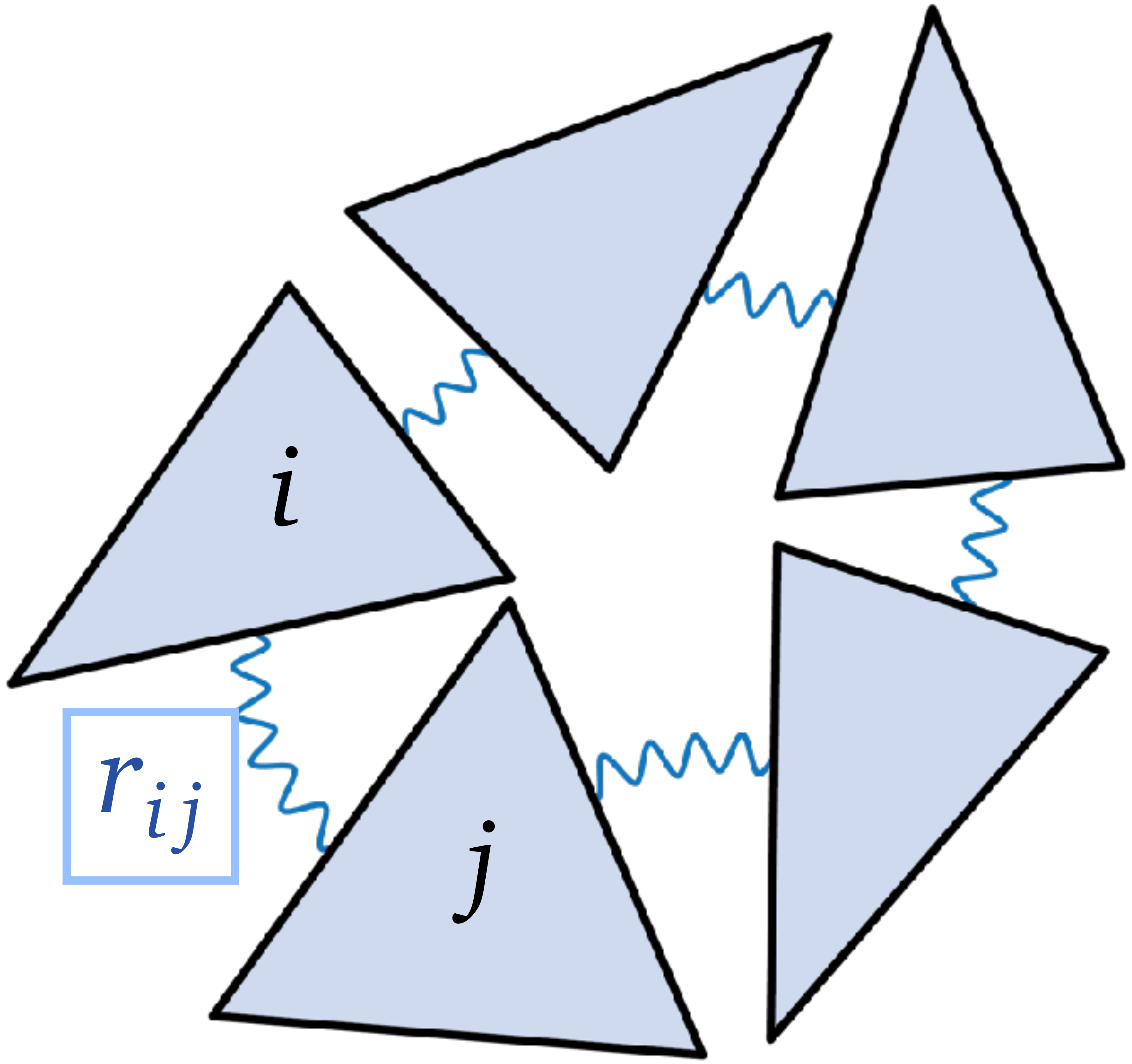

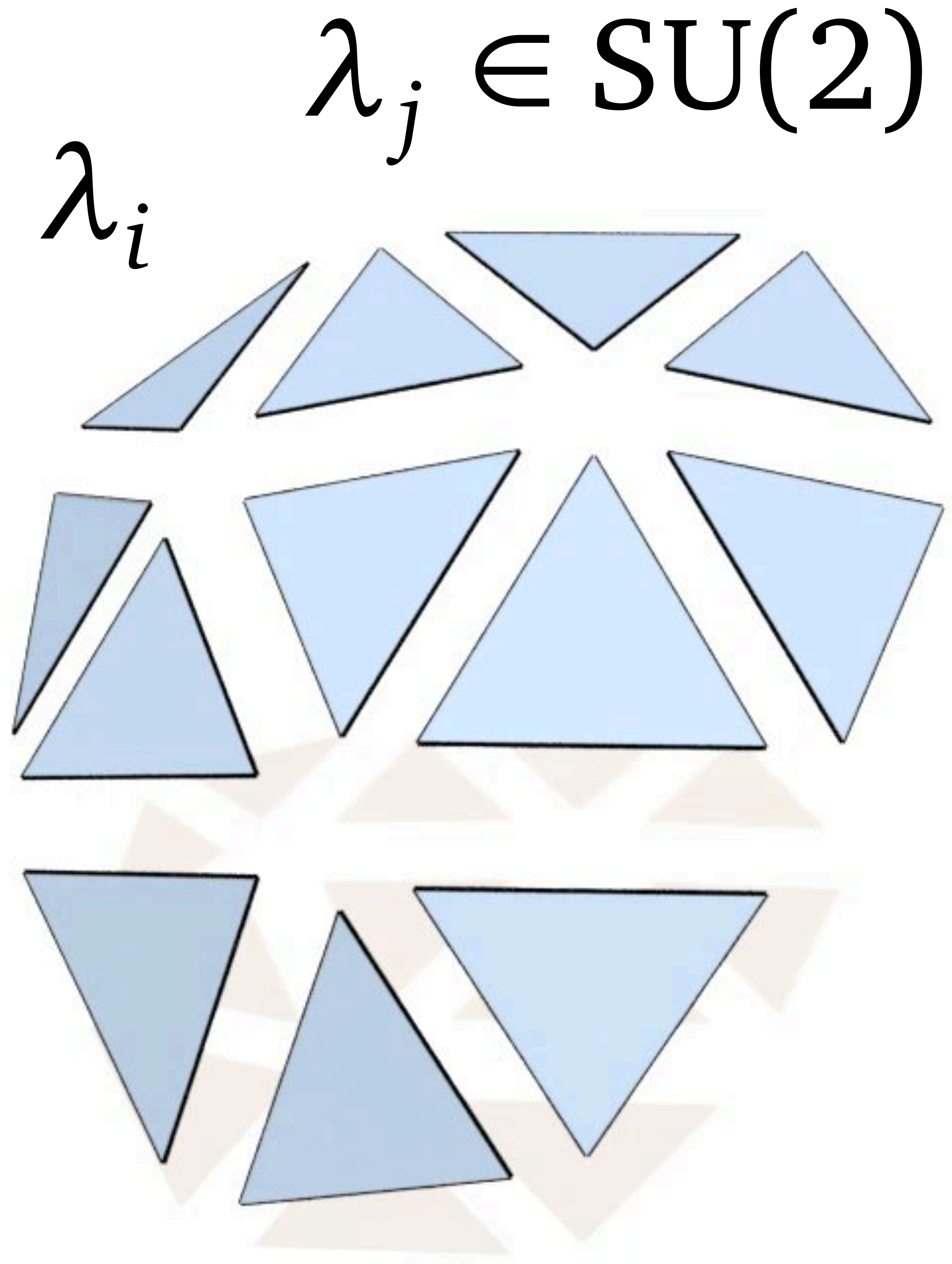

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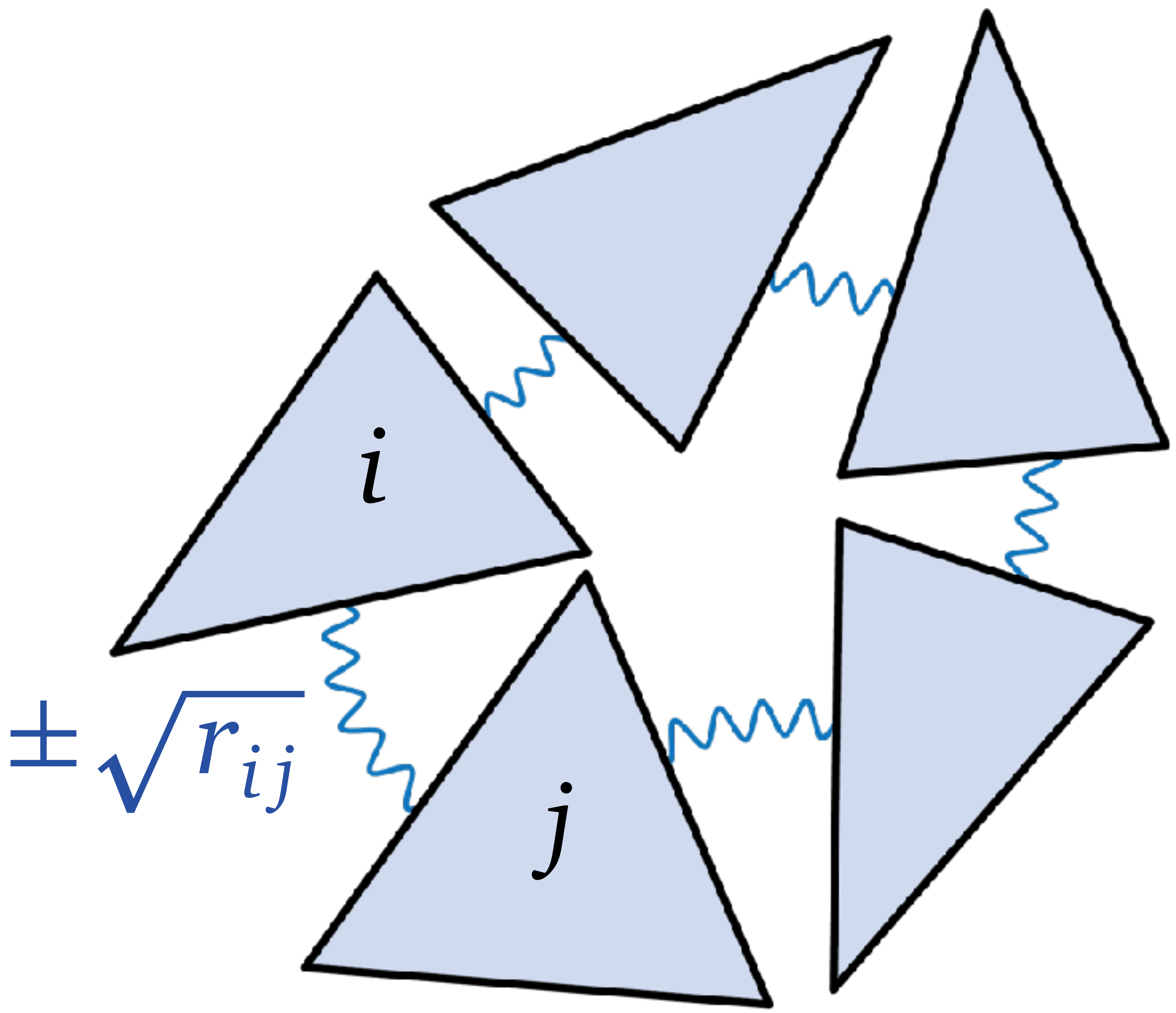
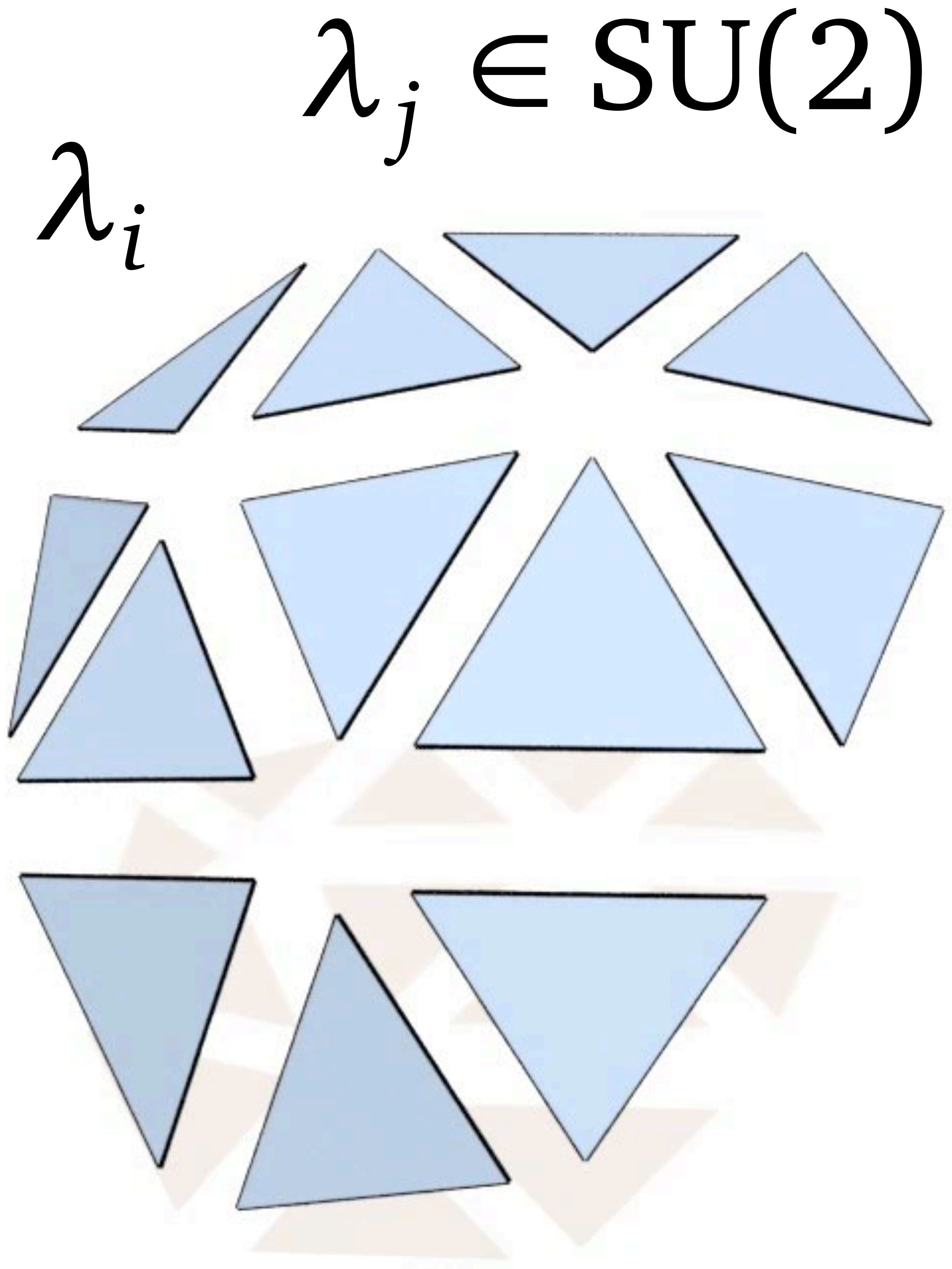
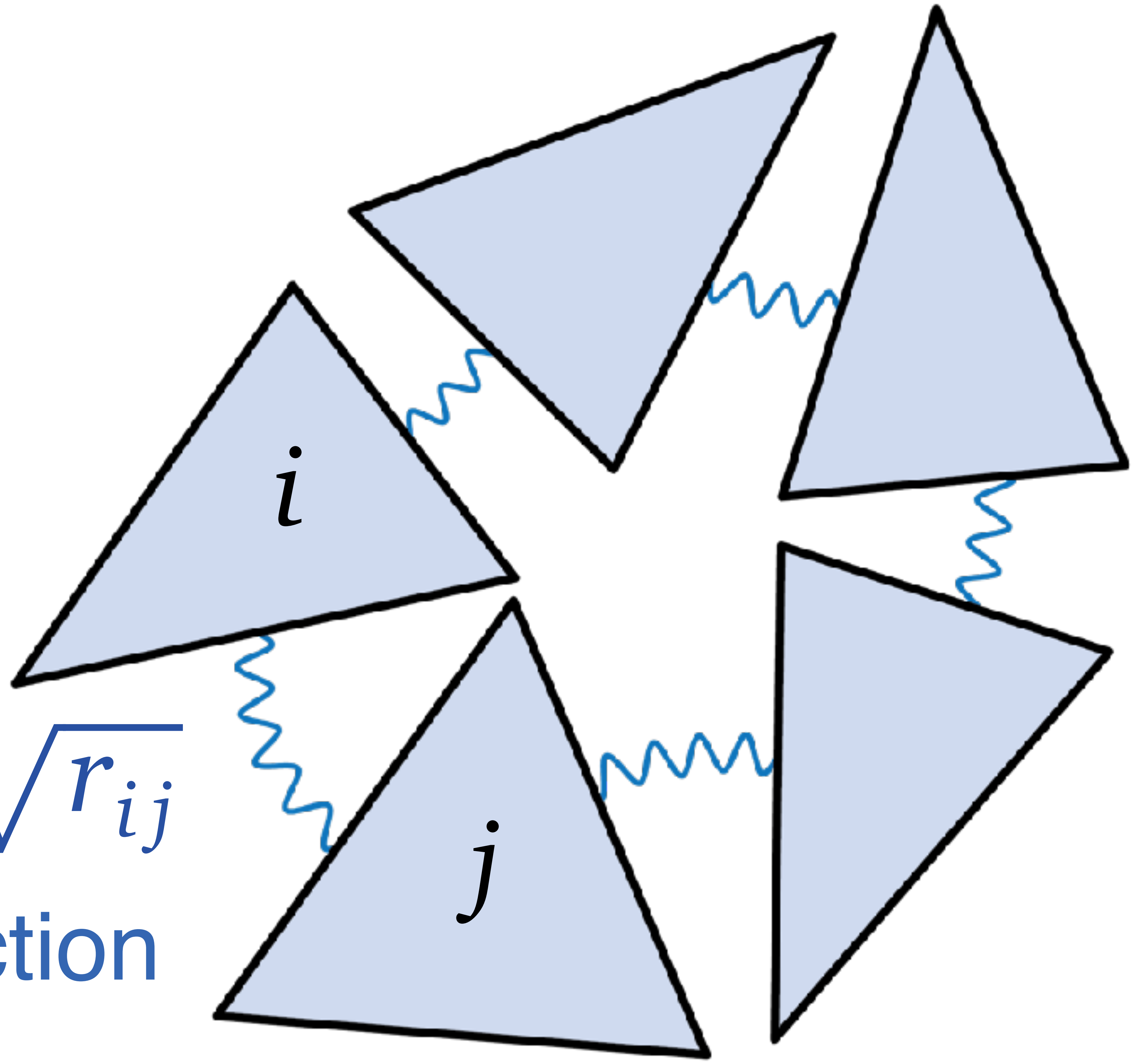
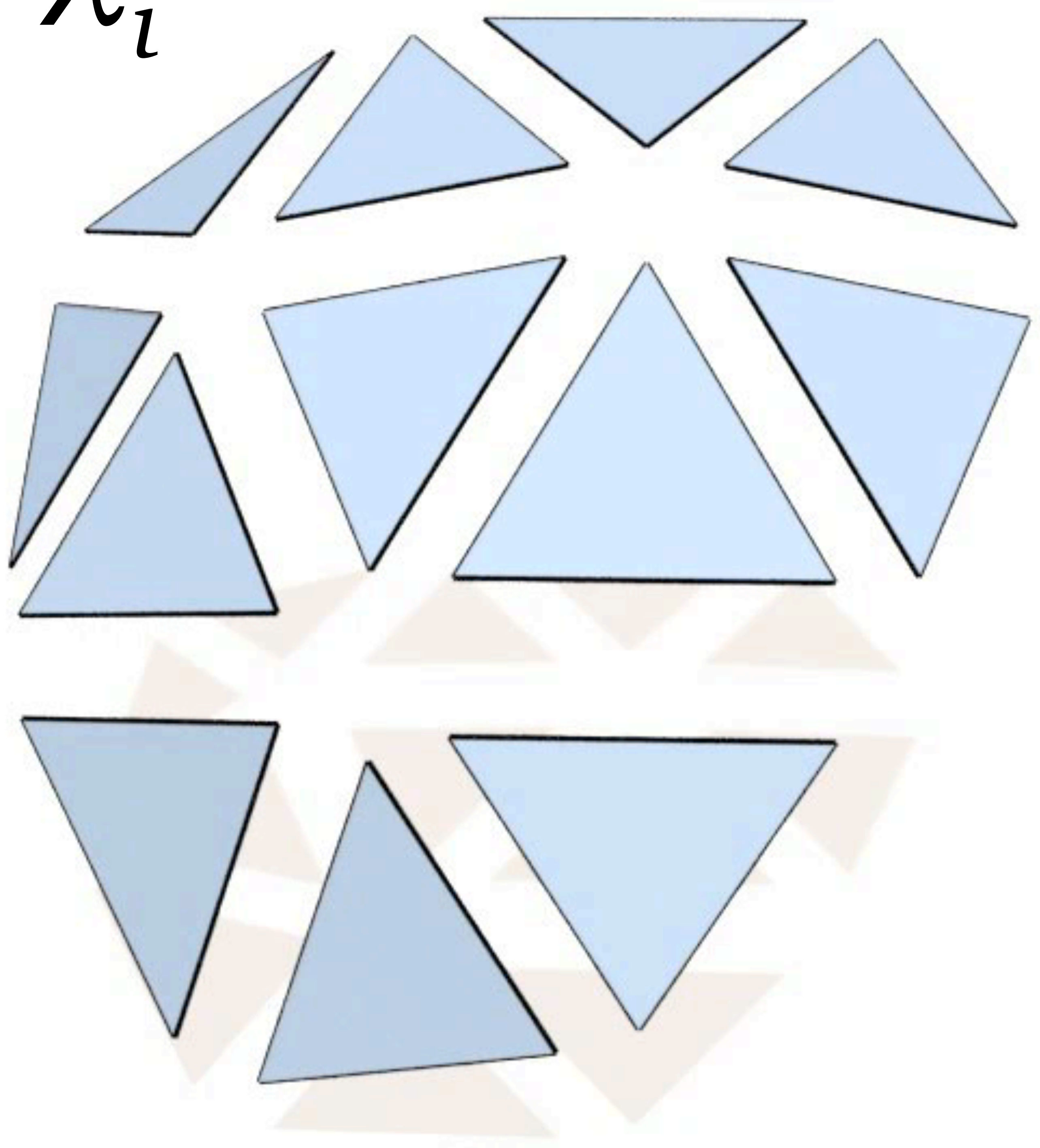




Lifting rotations to spinors





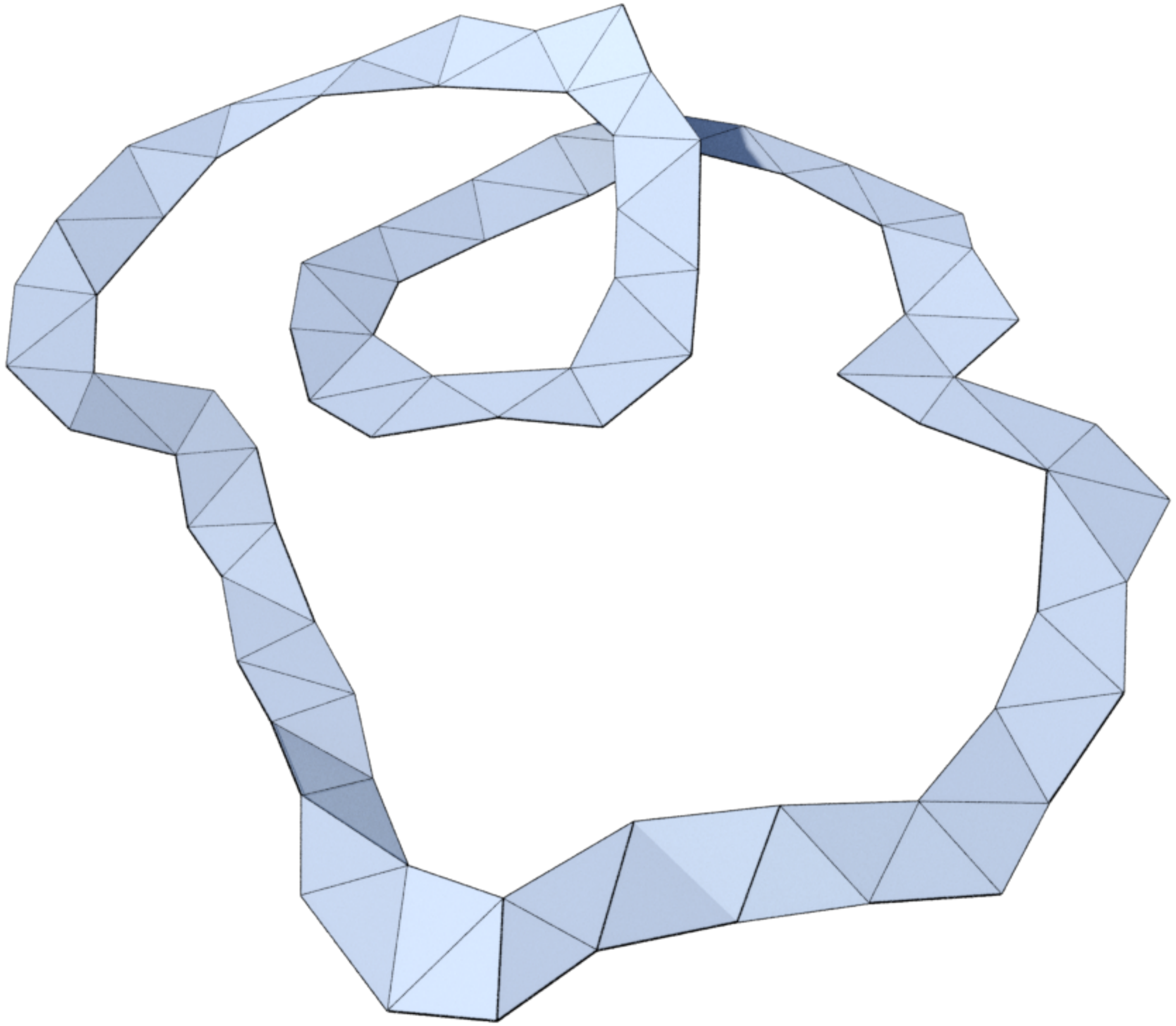
$\tau_{ij} := \pm\sqrt{r_{ij}}$

Spin connection

The sign encodes $\mathfrak{q}$

Gauss–Bonnet Theorem

Given $\gamma \in \{\text{closed strips}\}$

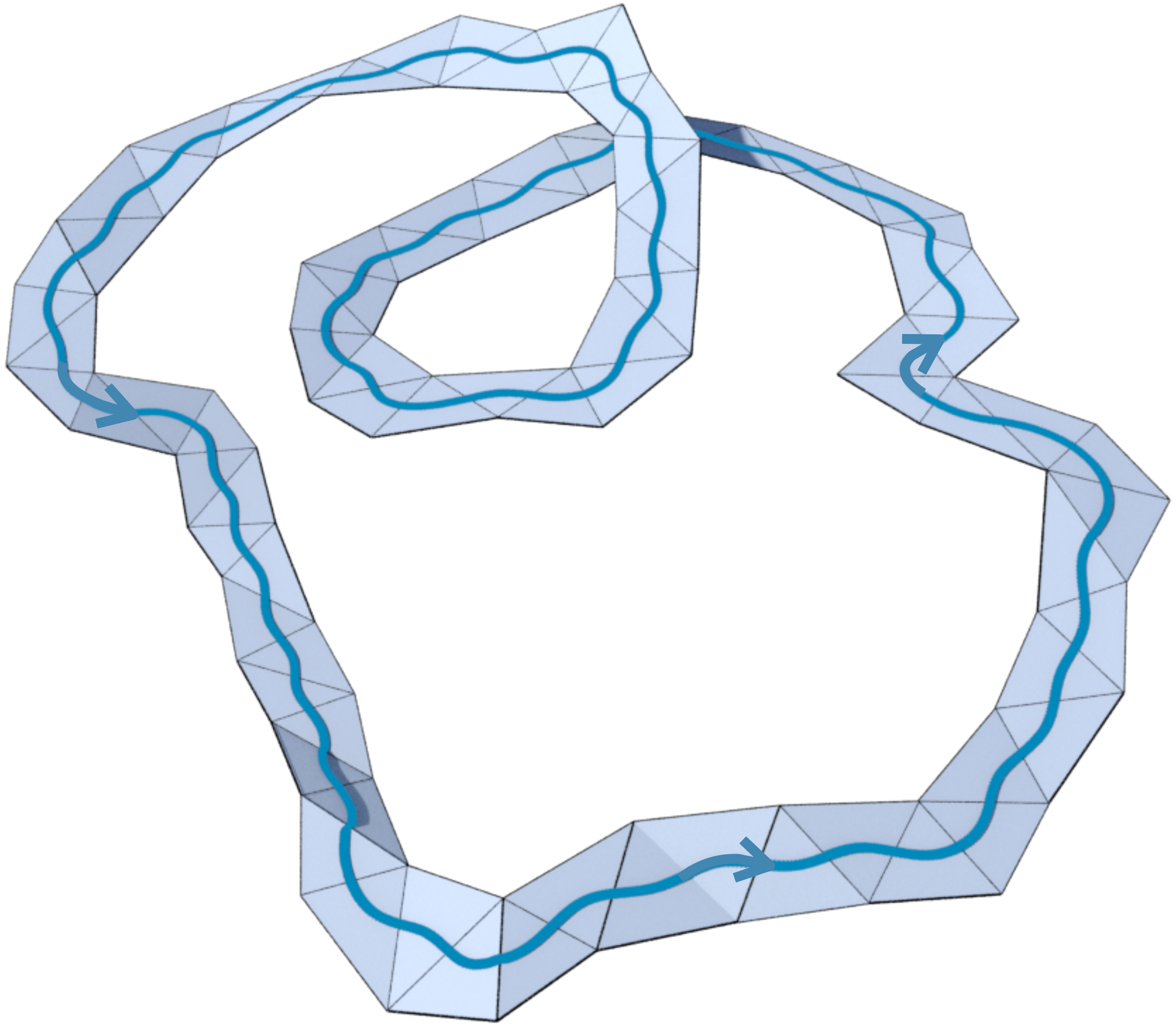


Gauss–Bonnet Theorem

Given $\gamma \in \{\text{closed strips}\}$

Represent it as a path

$$\hat{\gamma} : \mathbb{S}^1 \to M$$



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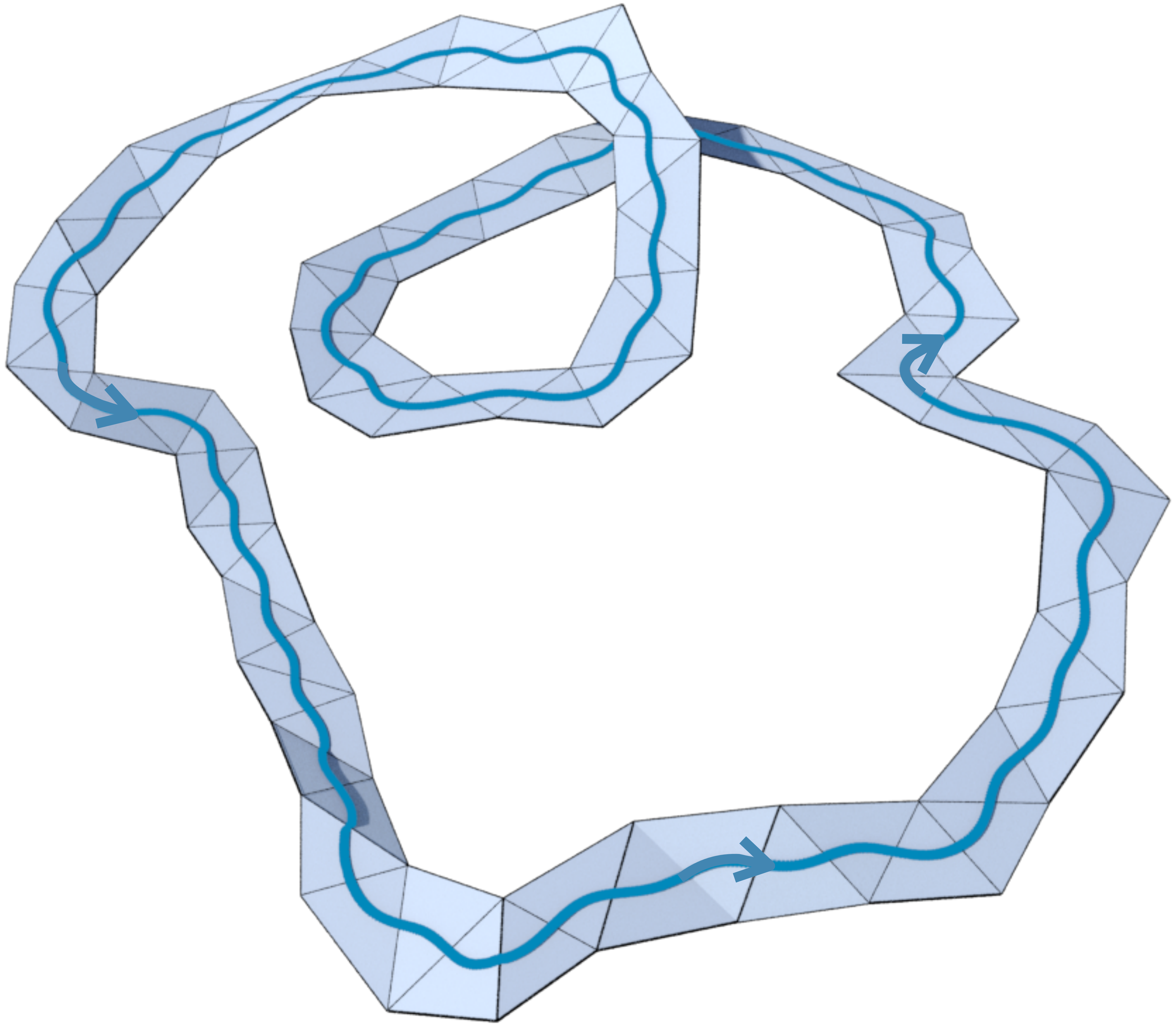
$$\prod_{\hat{\gamma}} r_{ij}$$



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Given $\gamma \in \{\text{closed strips}\}$

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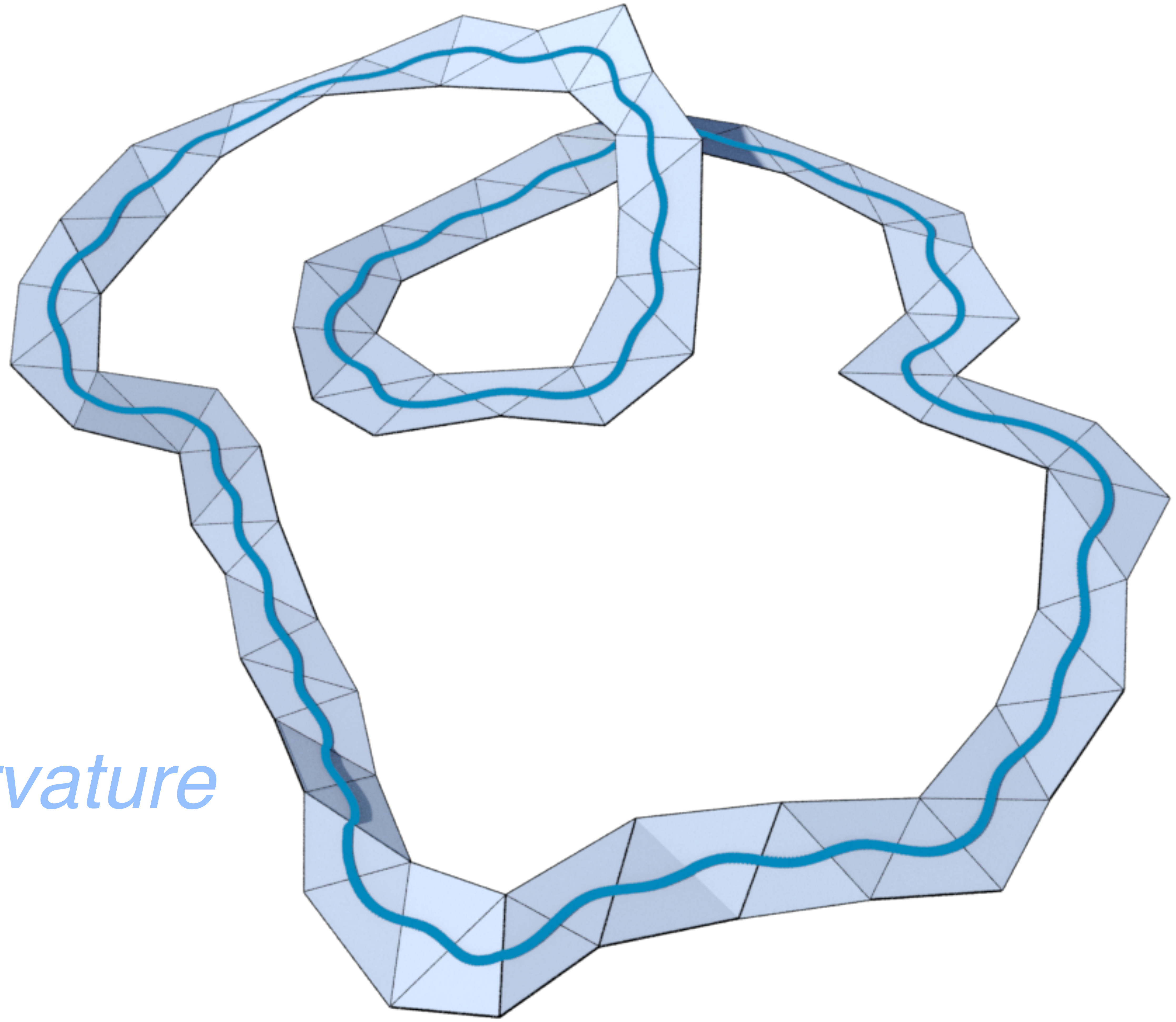
$$\hat{\gamma} : \mathbb{S}^1 \to M$$

$$\prod_{\hat{\gamma}} r_{ij} = \exp\left(2\pi\mathbf{i} - \mathbf{i}\int_{\hat{\gamma}} \boxed{\kappa_g}\right)$$

geodesic curvature



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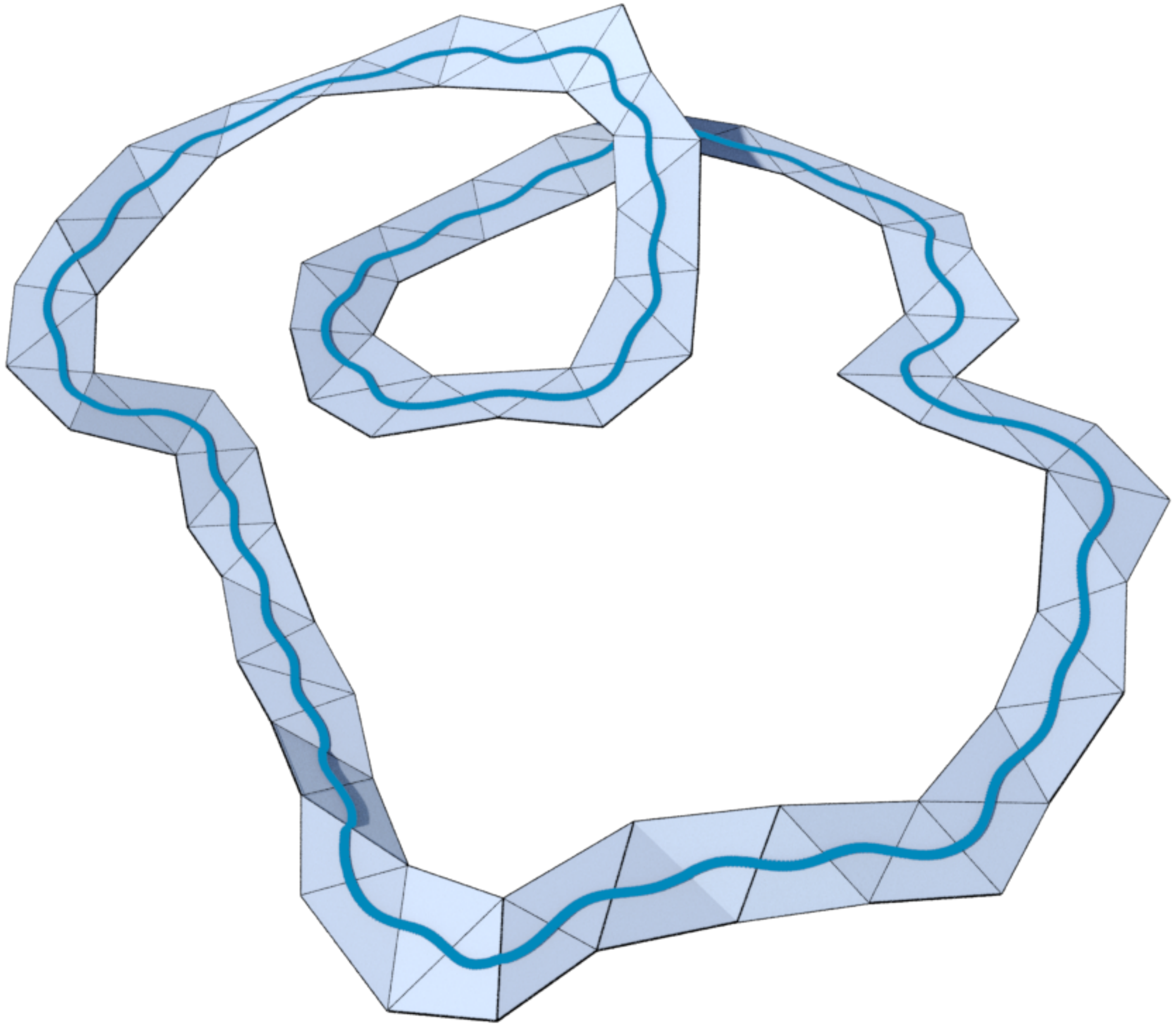


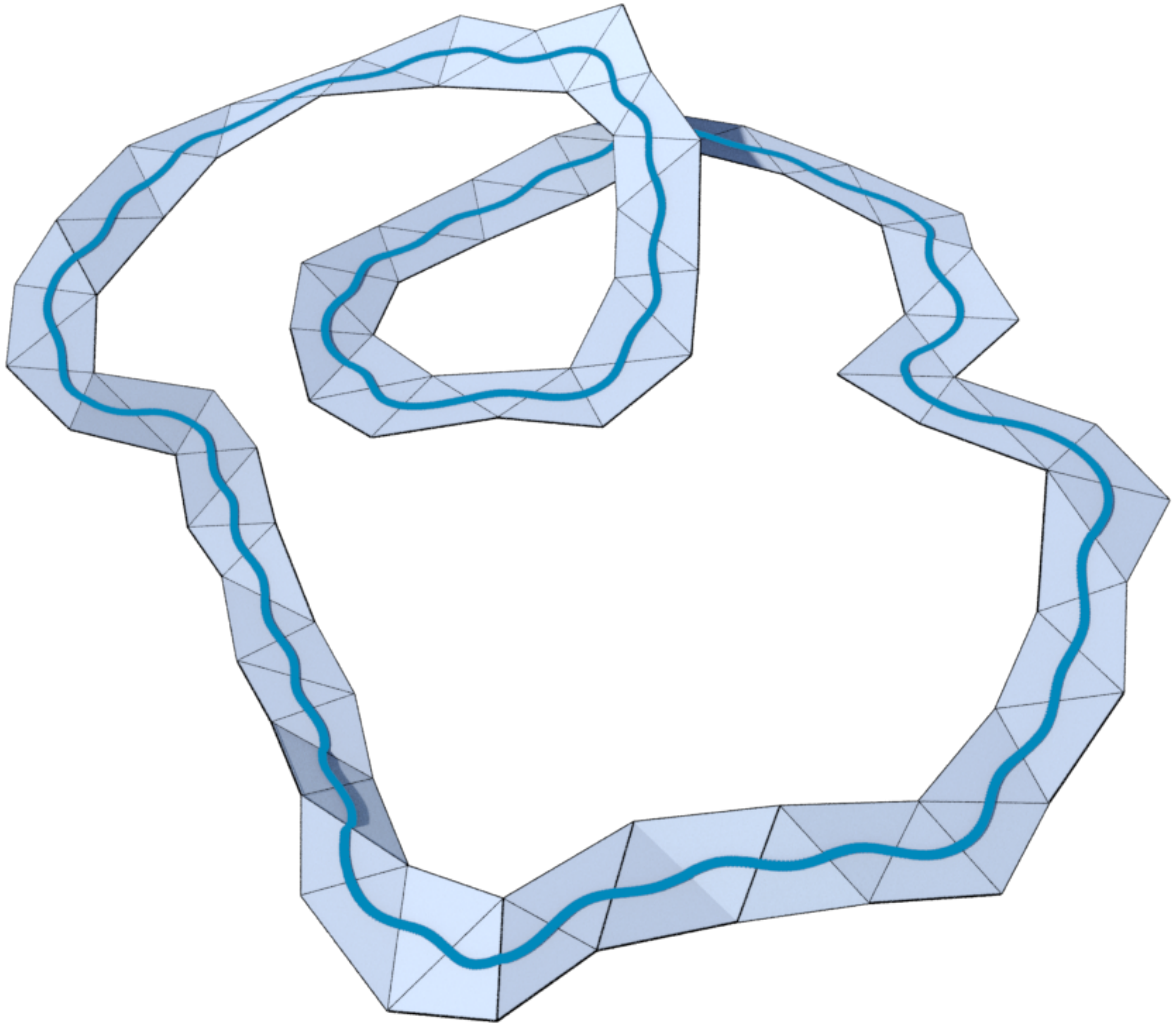
Spin Gauss–Bonnet Theorem

Given $\gamma \in \{\text{closed strips}\}$

Represent it as a path

$$\hat{\gamma} : \mathbb{S}^1 \to M$$

$$\prod_{\hat{\gamma}} \tau_{ij} = \exp\left(2\pi\mathbf{i} - \mathbf{i}\int_{\hat{\gamma}} \kappa_g\right)$$



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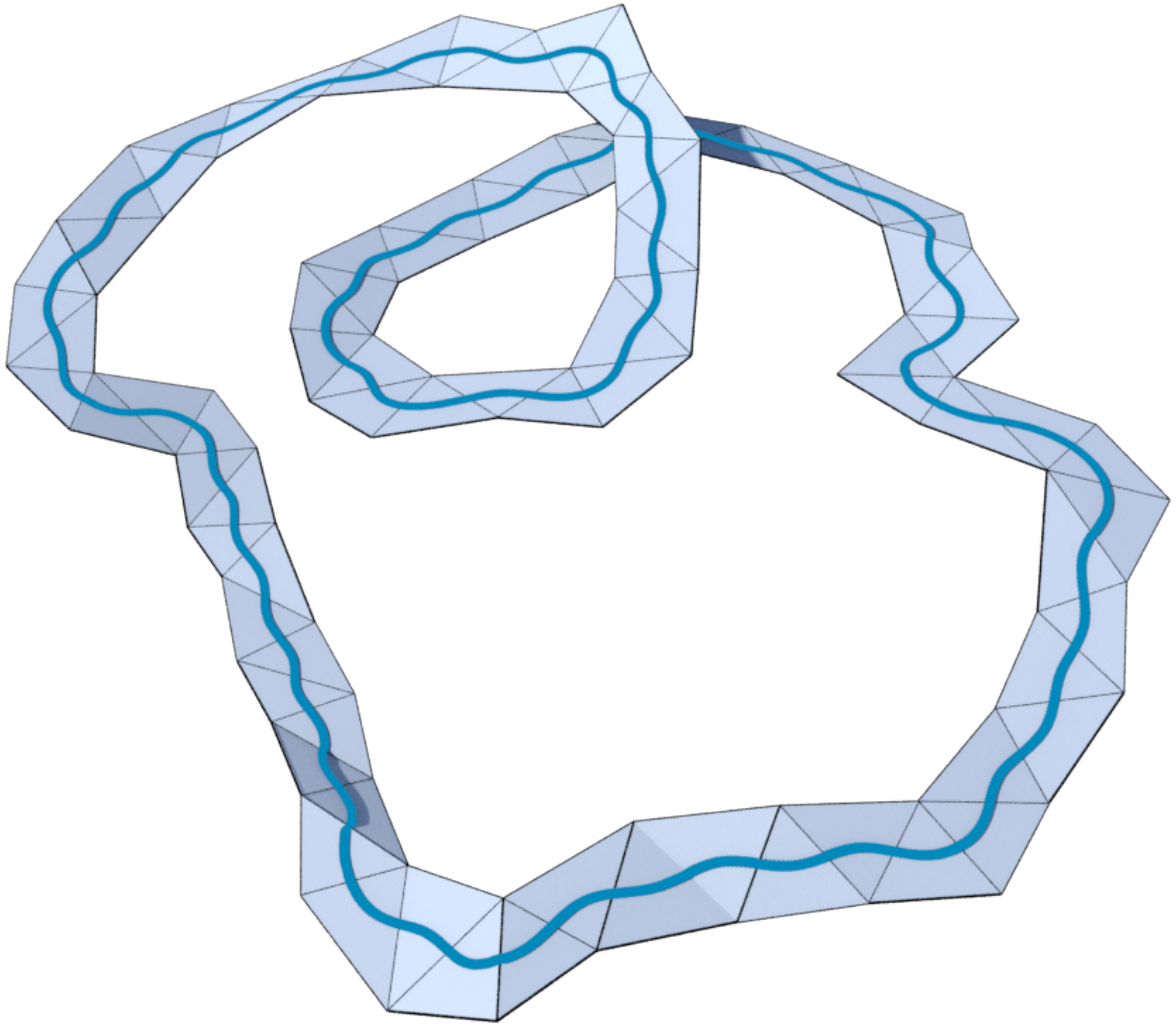


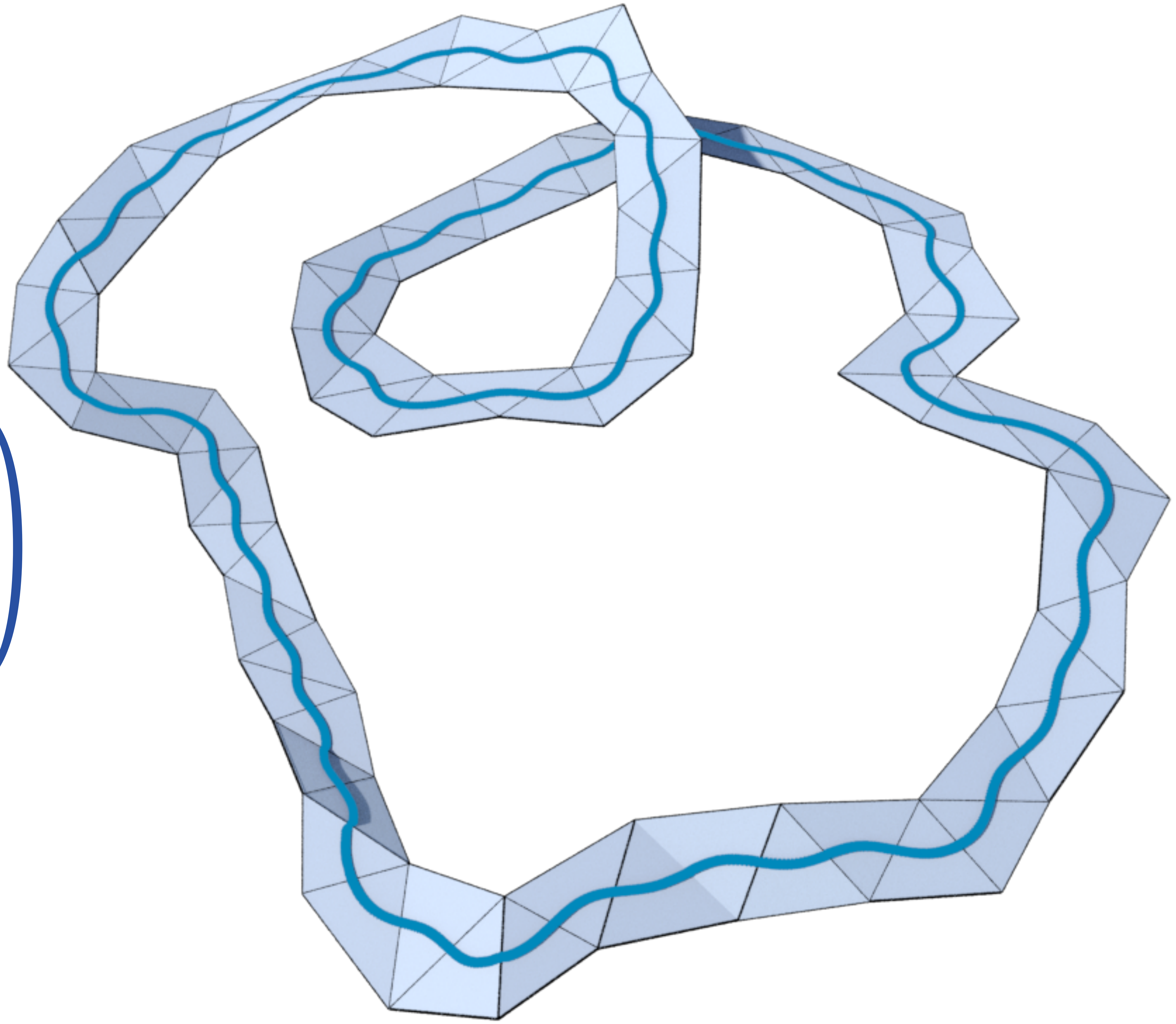
Spin Gauss–Bonnet Theorem

Given $\gamma \in \{\text{closed strips}\}$

Represent it as a path

$$\hat{\gamma} : \mathbb{S}^1 \to M$$

$$\prod_{\hat{\gamma}} \tau_{ij} = \pm \exp\left(\pi \mathbf{i} - \frac{\mathbf{i} \int_{\hat{\gamma}} \kappa_g}{2}\right)$$



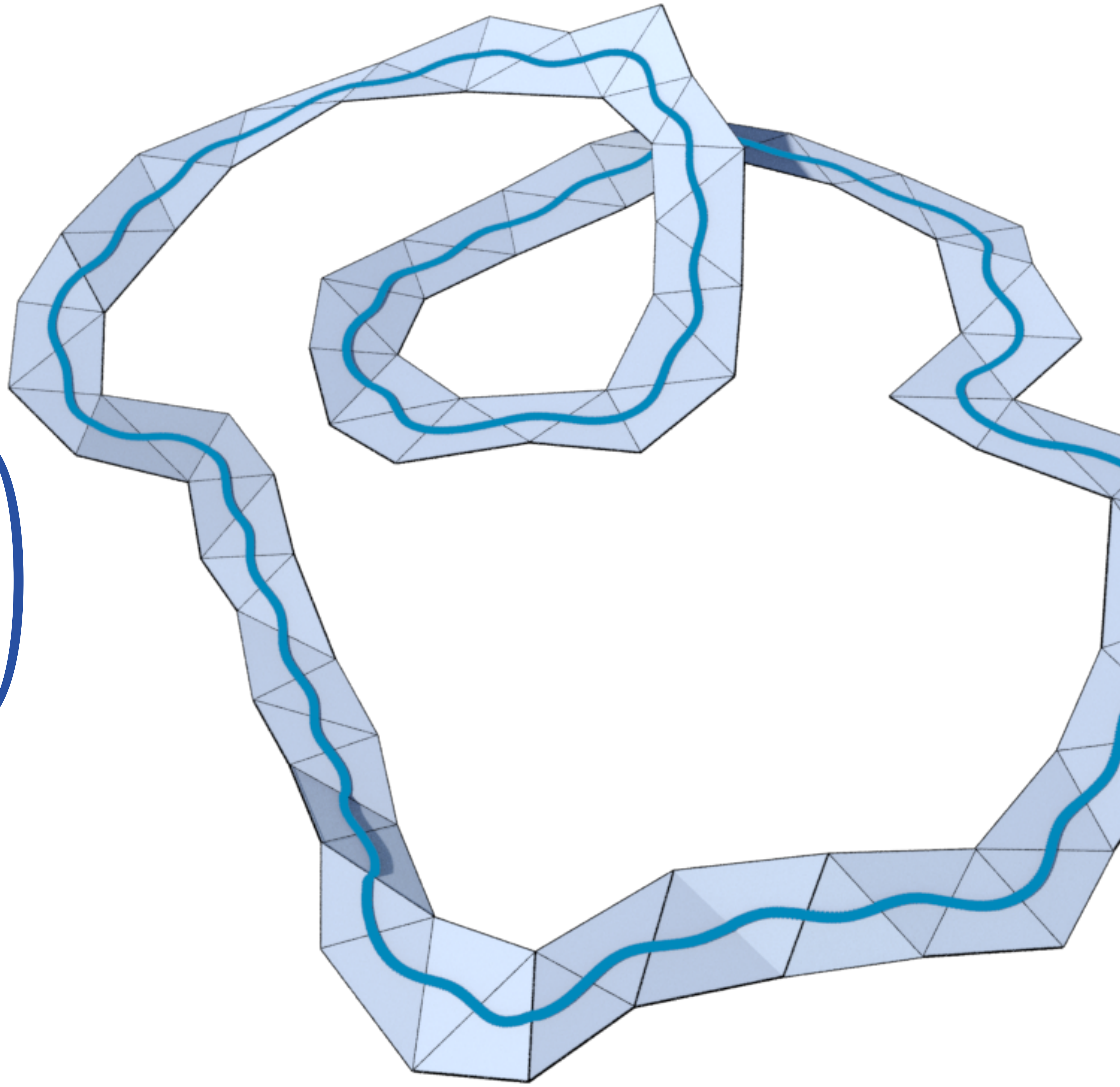
Spin Gauss–Bonnet Theorem

Given $\gamma \in \{\text{closed strips}\}$

Represent it as a path

$$\hat{\gamma} : \mathbb{S}^1 \to M$$

$$\prod_{\hat{\gamma}} \tau_{ij} = (-1)^{\mathfrak{q}_\tau(\gamma)} \exp\left(\pi \mathbf{i} - \frac{\mathbf{i} \int_{\hat{\gamma}} \kappa_g}{2}\right)$$



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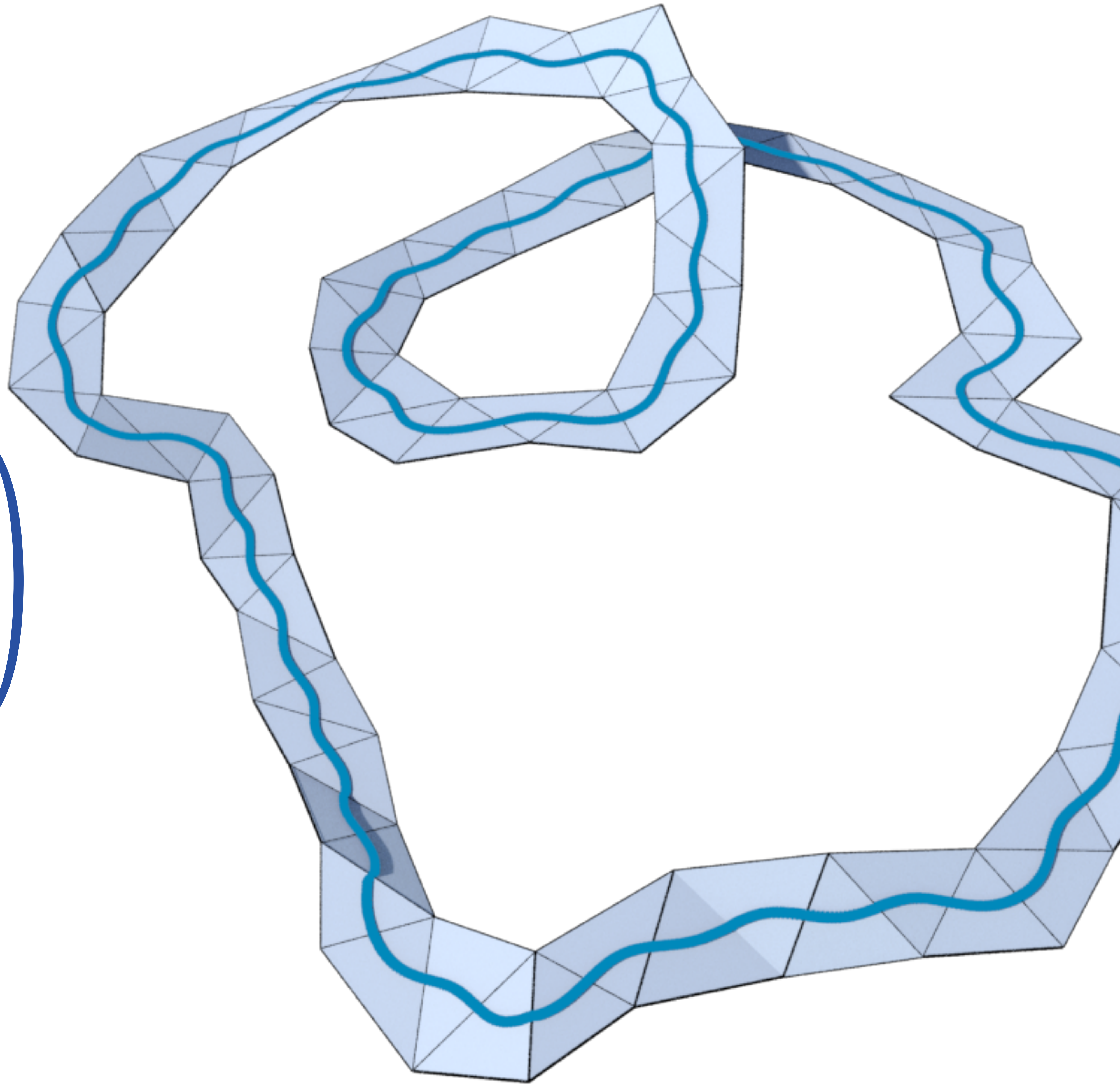
$$\mathfrak{q}_\tau : \{\text{closed strips}\} \to \mathbb{Z}_2$$



Spin Structure

Theorem

$\mathfrak{q}_\tau : \{\text{closed strips}\} \to \mathbb{Z}_2$ *is a quadratic form associated with* $[\cdot \cap \cdot]$.

Spin Structure

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Spin Structure

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Given M with a $\boxed{\text{desired metric}}$ and $\boxed{\text{Figure-8/0 configuration } \mathfrak{q}}$

Rotational connection r_{ij}

Spin connection $\tau_{ij} = \pm_{ij}\sqrt{r_{ij}}$ so that $\mathfrak{q}_\tau = \mathfrak{q}$

Emergent surface

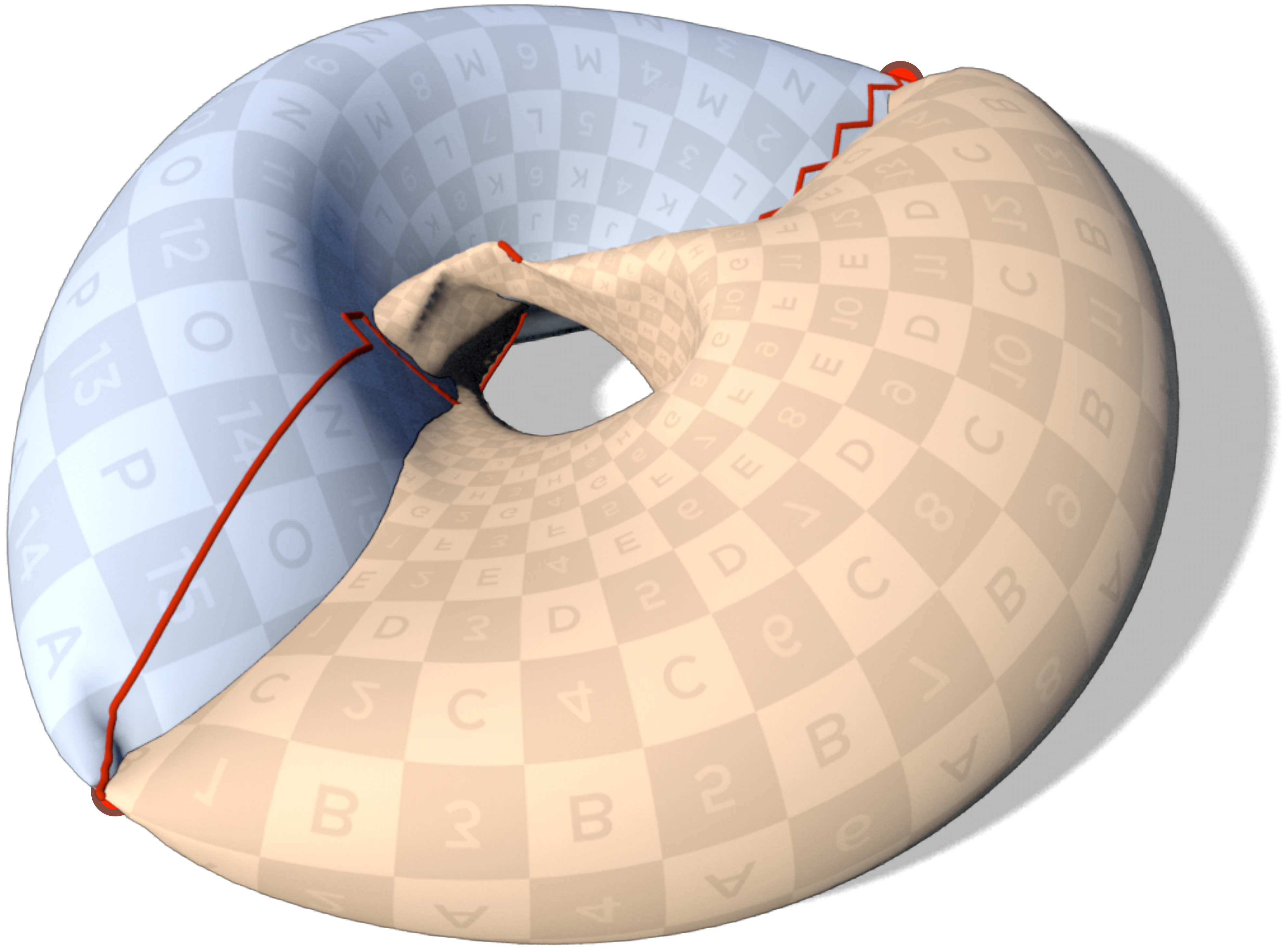
Microscopic scale

Setting up gauge field r_{ij}
and a quadratic form $\mathfrak{q}$

Macroscopic scale

minimize $$\sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_\epsilon^2$$

minimize $|\mathfrak{s}| = |\mathfrak{q}_f - \mathfrak{q}|$



Emergent surface

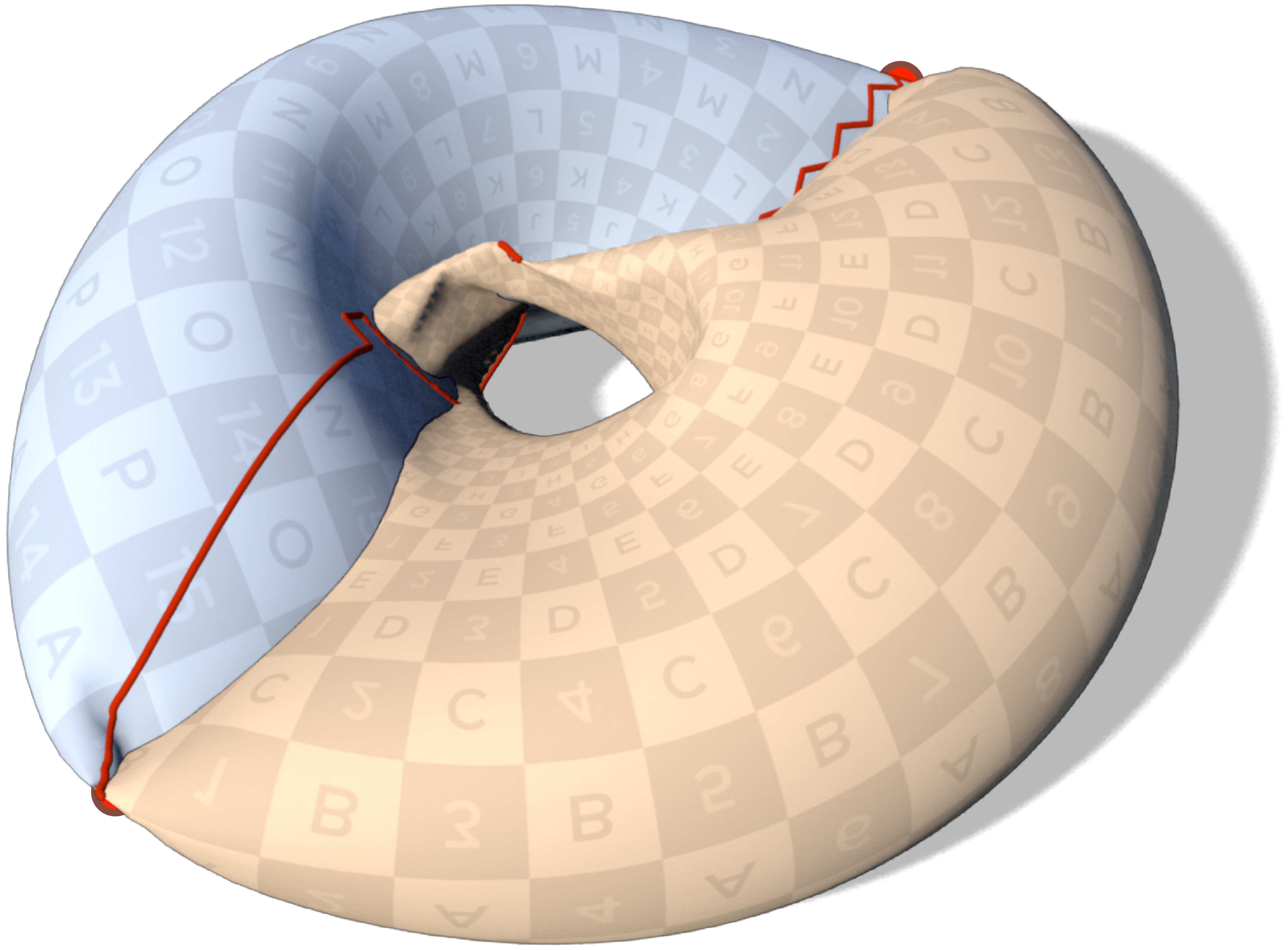
Microscopic scale

Spin connection τ_{ij}

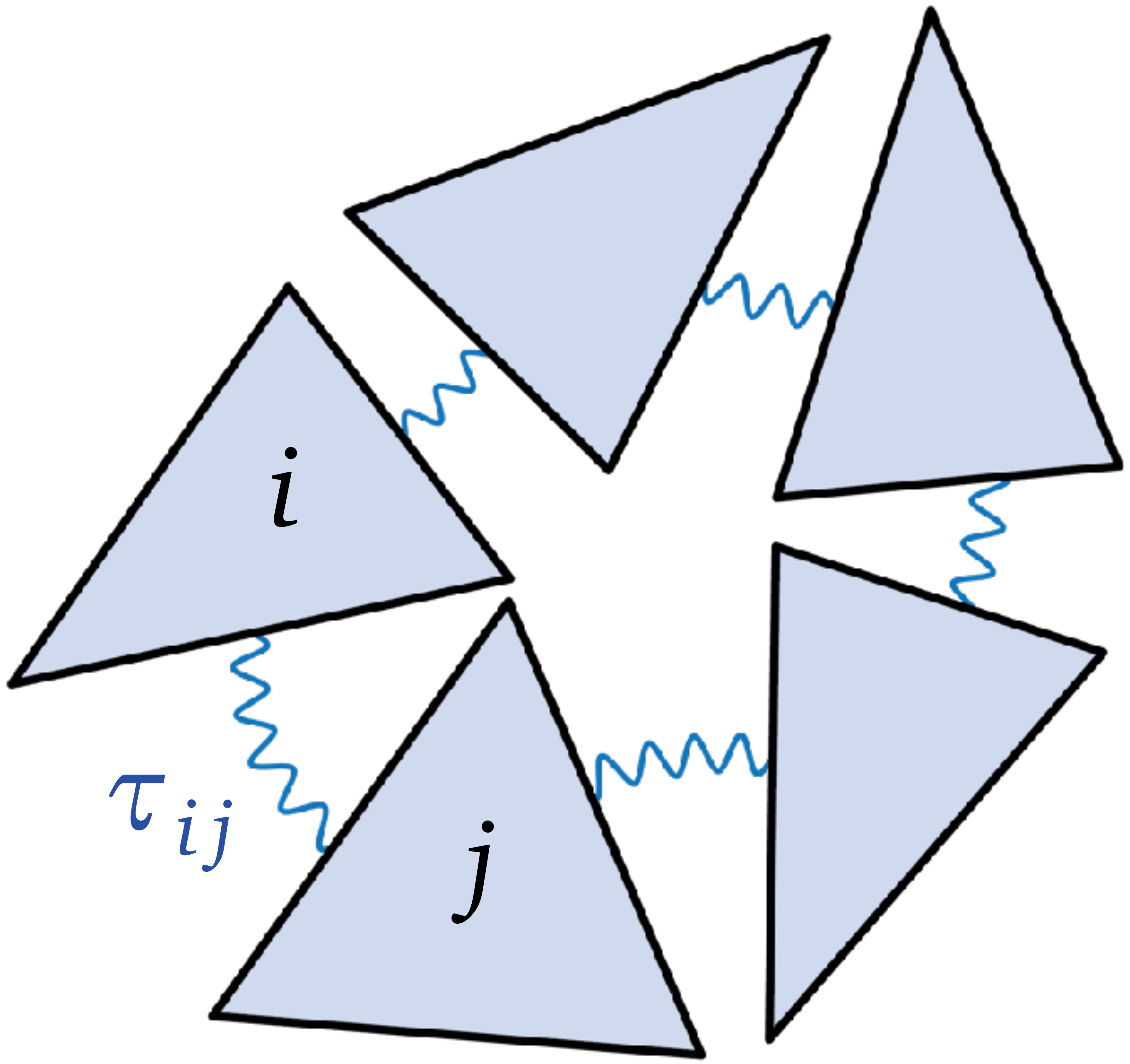
Macroscopic scale

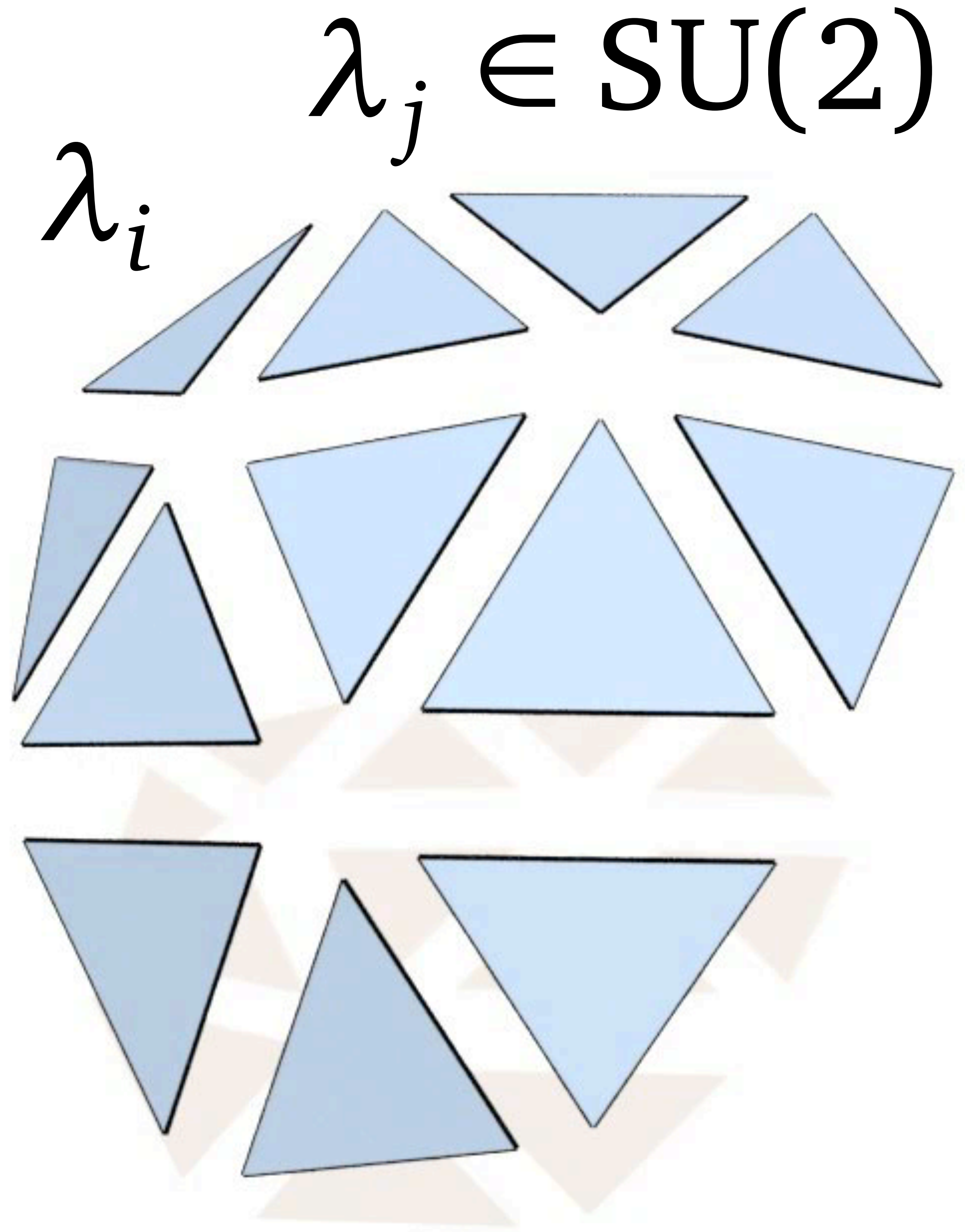
$$\text{minimize} \sum_{\text{all edges}} \left| Q_j - Q_i \circ r_{ij} \right|_\epsilon^2$$

$$\text{minimize } |\mathfrak{s}| = |\mathfrak{q}_f - \mathfrak{q}|$$



Rim Representation





We can use the spin connection τ_{ij} to measure whether λ_i, λ_j have consistent chosen signs.

Rim Representation

Theorem [C., Knöppel, Pinkall, Schröder 2018]
Let $f : M \longrightarrow \mathbb{R}^3$ *be a non-degenerate triangular surface,*
$Q_i \in \mathrm{SO}(3)$ *be the rotation part of* $(df)_i$ *(polar decomposition),*
$\lambda_i \in \mathrm{SU}(2)$ *be any unit quaternion that "squares" to* Q_i.

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Across neighboring triangles, measure the signature

$$(-1)^{\mathfrak{s}_{ij}} := \mathrm{sgn}\langle \lambda_j, \lambda_i \circ \tau_{ij} \rangle_{\mathbb{R}^4}$$

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Then the rimmed surface $(f, \mathfrak{s})$ *has the desired figure-8/0 property*

$$\mathfrak{q}_\tau = \mathfrak{q}_{(f,\mathfrak{s})}$$

Rim Representation

$$(-1)^{\mathfrak{s}_{ij}} := \operatorname{sgn}\langle \lambda_j, \lambda_i \circ \tau_{ij} \rangle_{\mathbb{R}^4}$$

$$|\mathfrak{s}| \leq \frac{1}{2} \sum_{\text{all edges}} \left| \lambda_j - \lambda_i \circ \tau_{ij} \right|^2$$

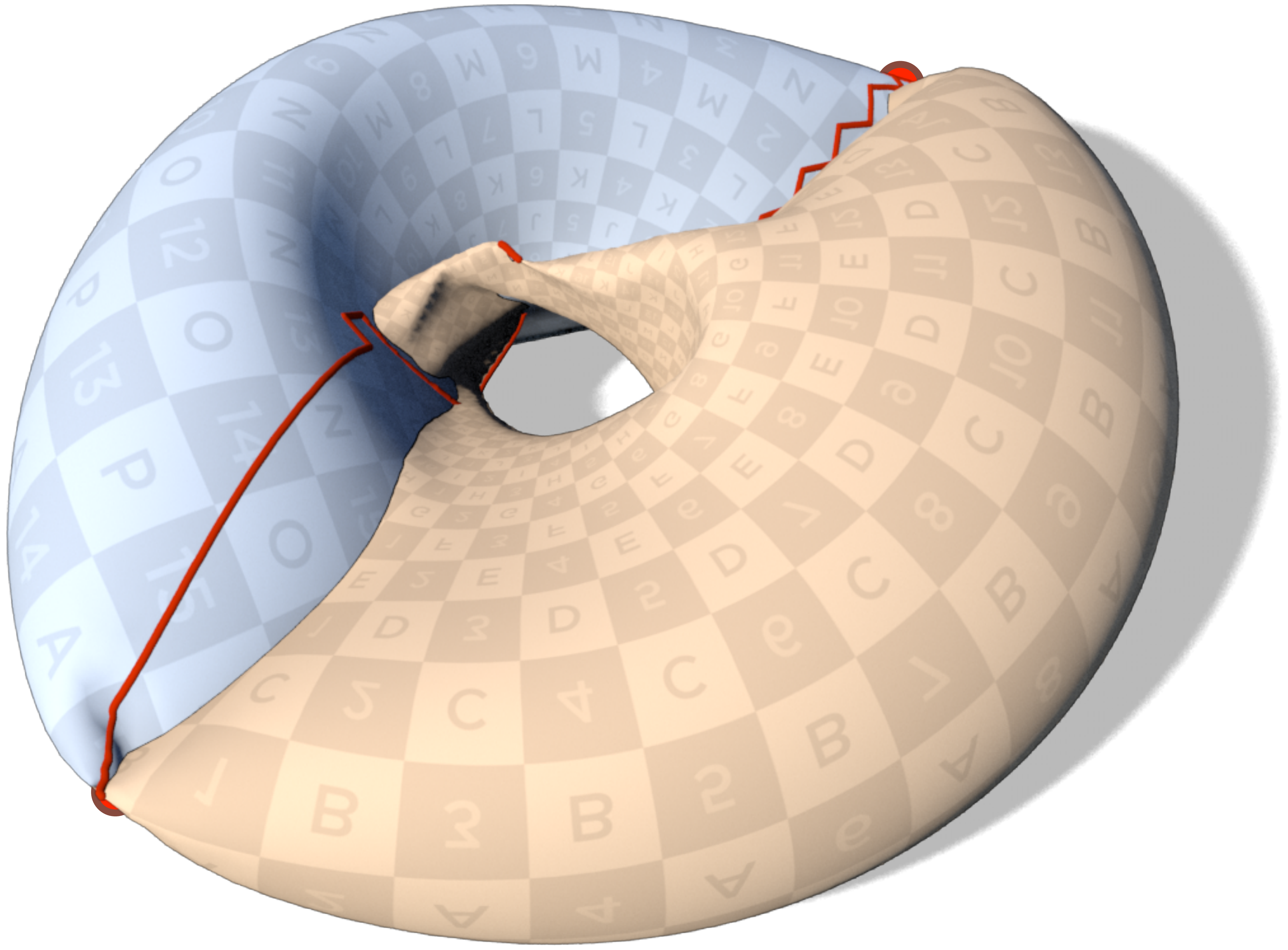
Emergent surface

Microscopic scale

Spin connection τ_{ij}

Macroscopic scale

minimize $\displaystyle\sum_{\text{all edges}} \left|\lambda_j - \lambda_i \circ \tau_{ij}\right|_{\epsilon}^{2}$



Emergent surface

Microscopic scale

Spin connection $\boxed{\tau_{ij}}$ *gauge field encodes*

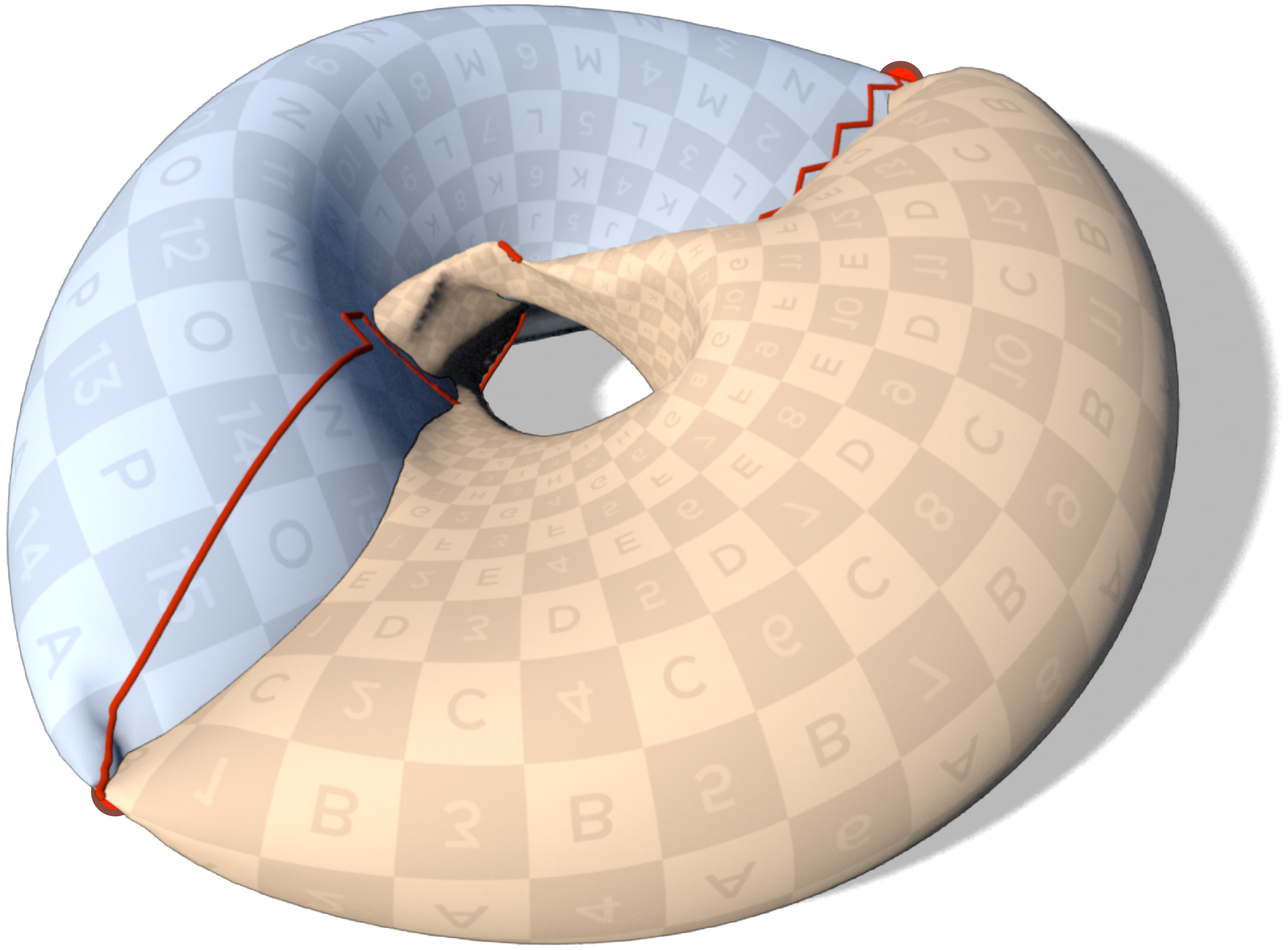
- *metric*
- *figure-8/0*

Macroscopic scale

$$\text{minimize} \sum_{\text{all edges}} \left| \boxed{\lambda_j} - \boxed{\lambda_i} \circ \tau_{ij} \right|_\epsilon^2$$

spinor field encodes

- *rotation field (frames)*
- *rims*



Emergent surface

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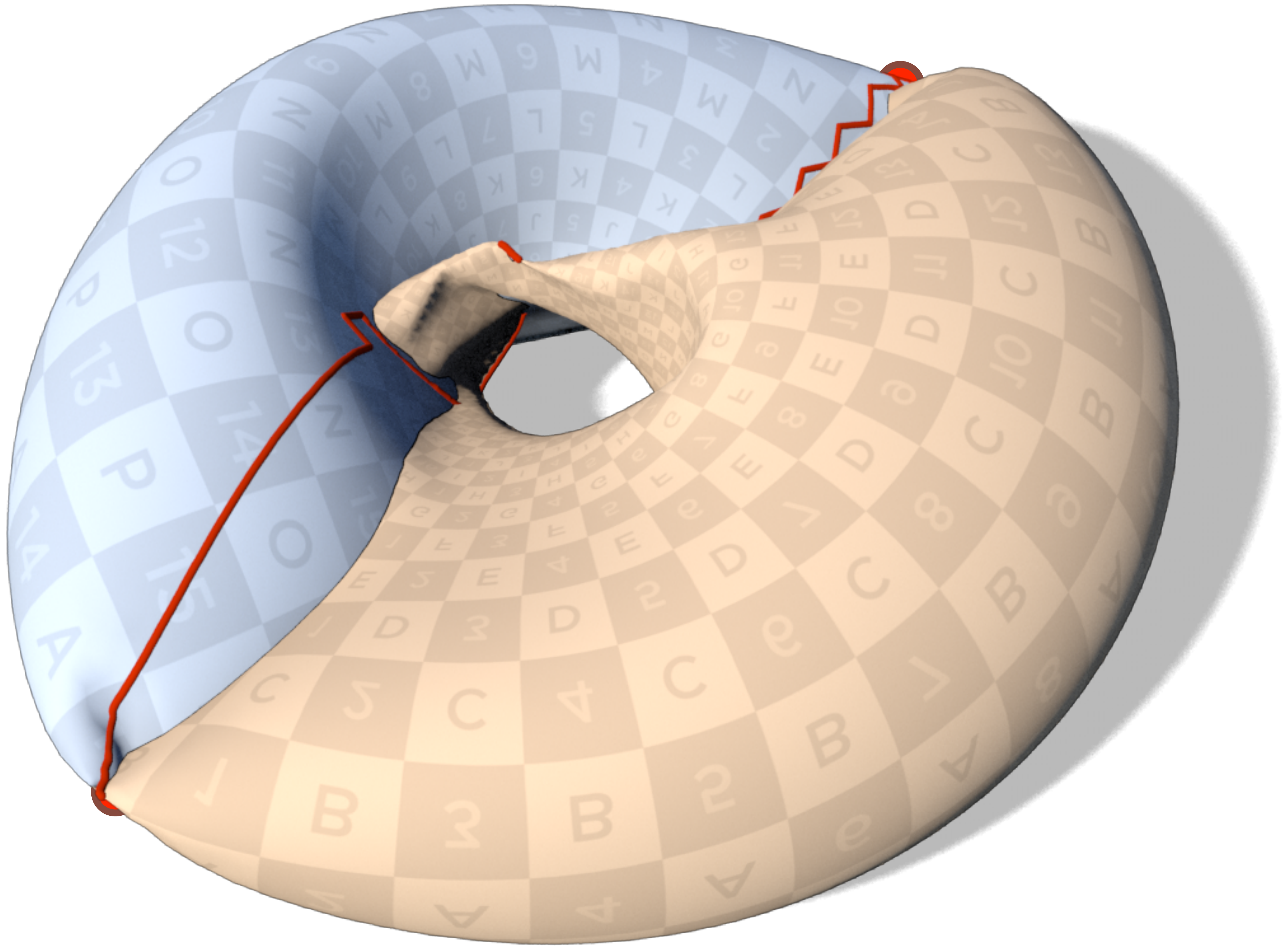

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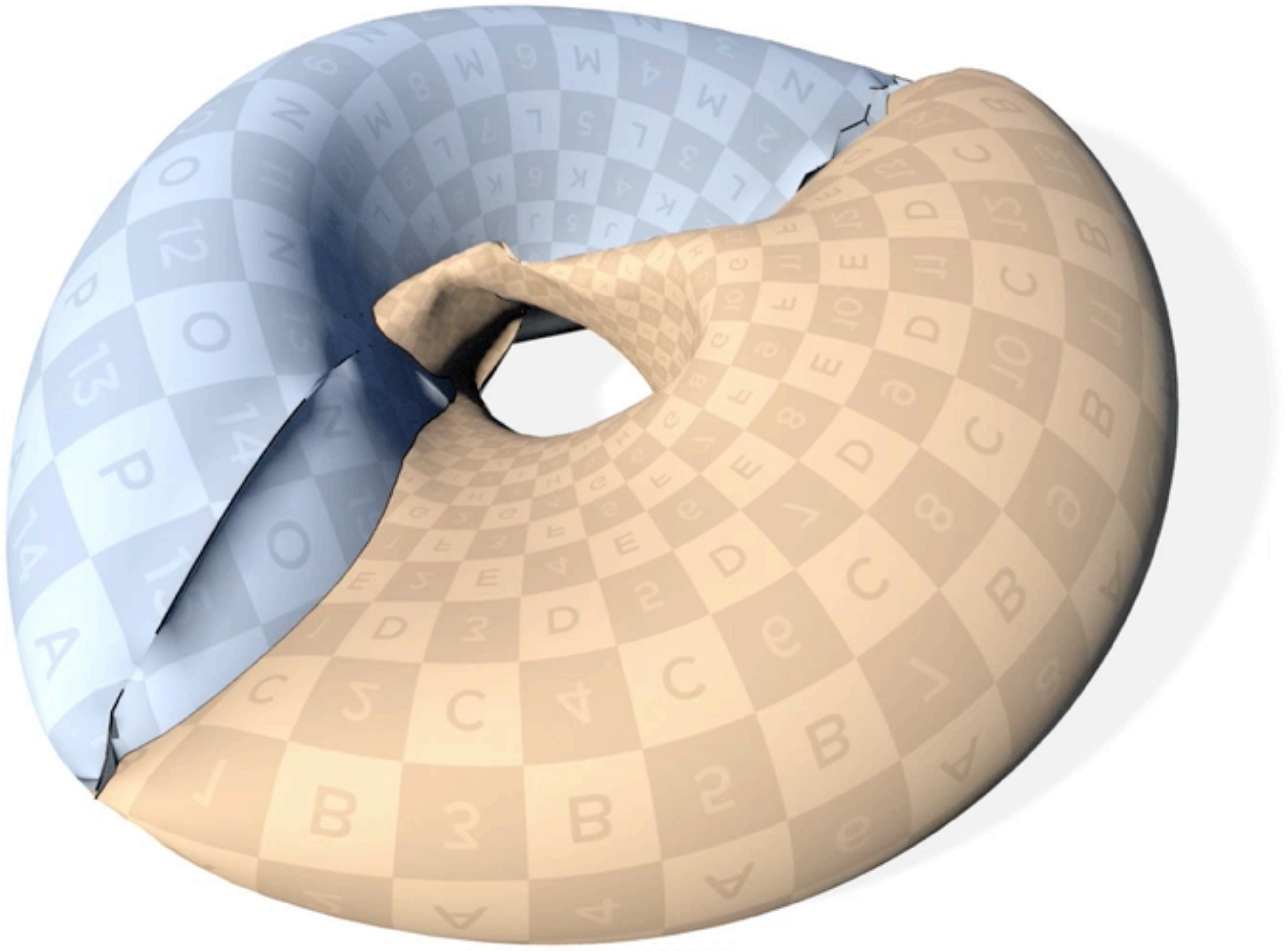
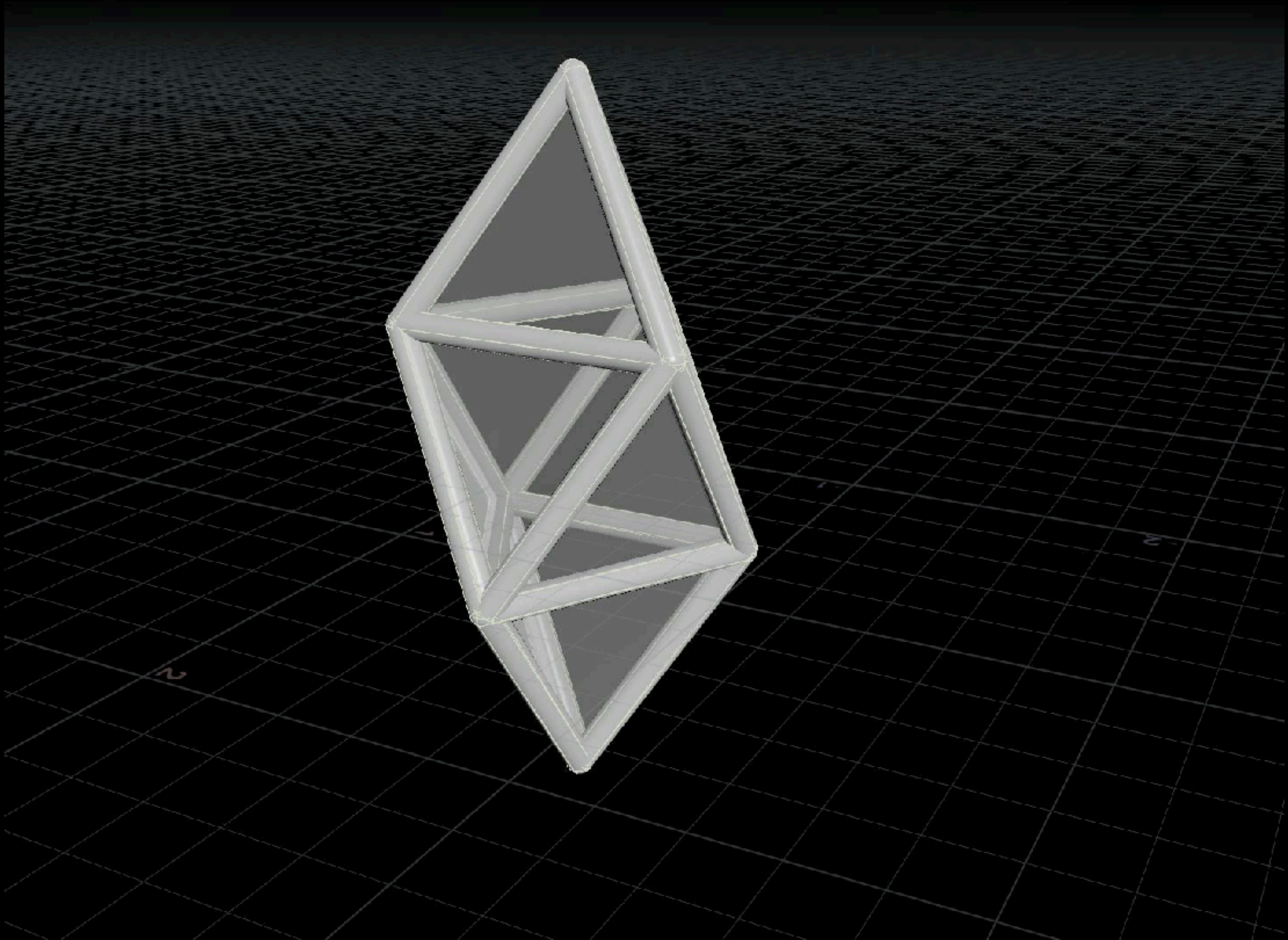
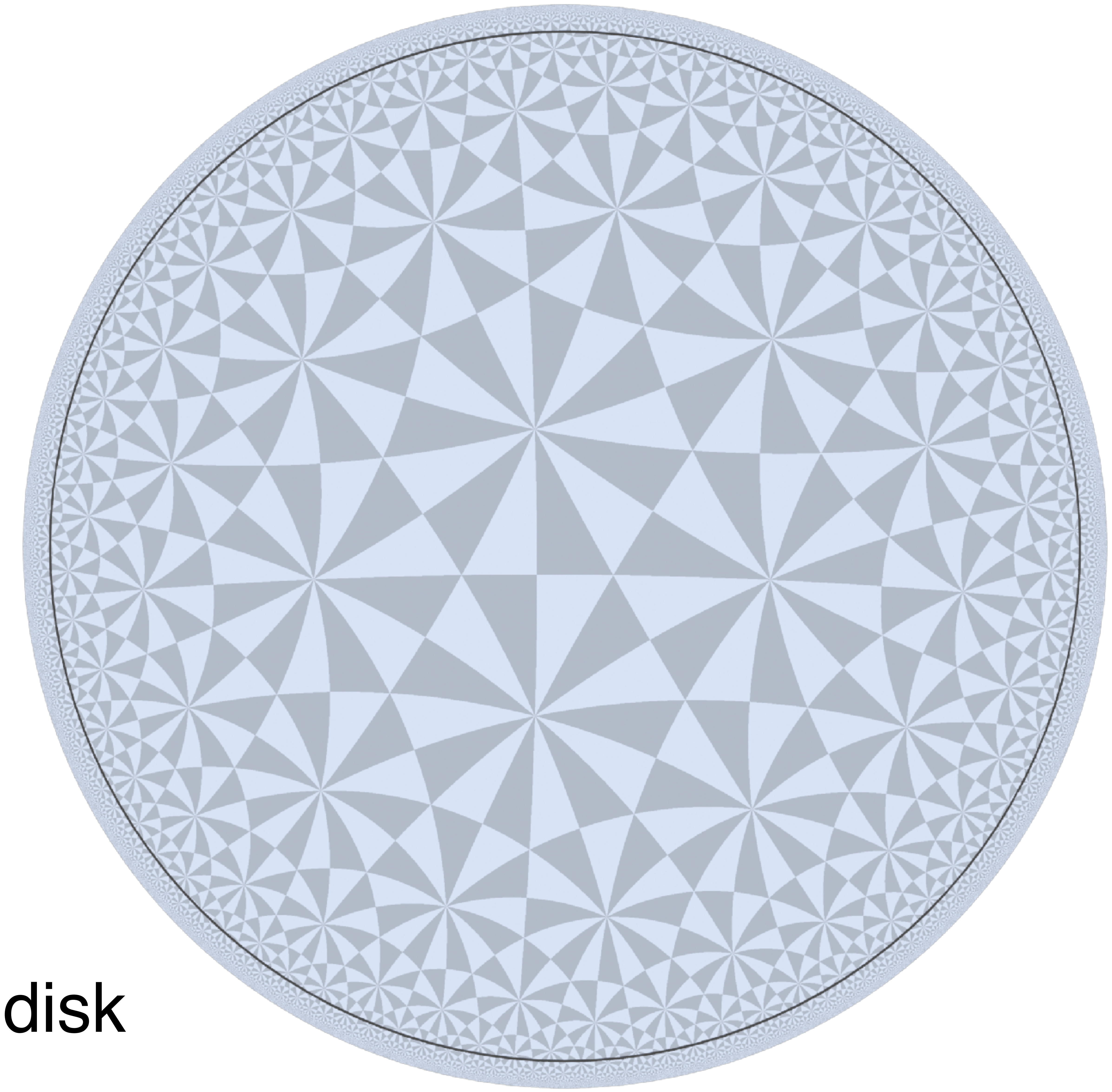


Pinch point resolved



A Disk in Hyperbolic Plane



Hyperbolic disk

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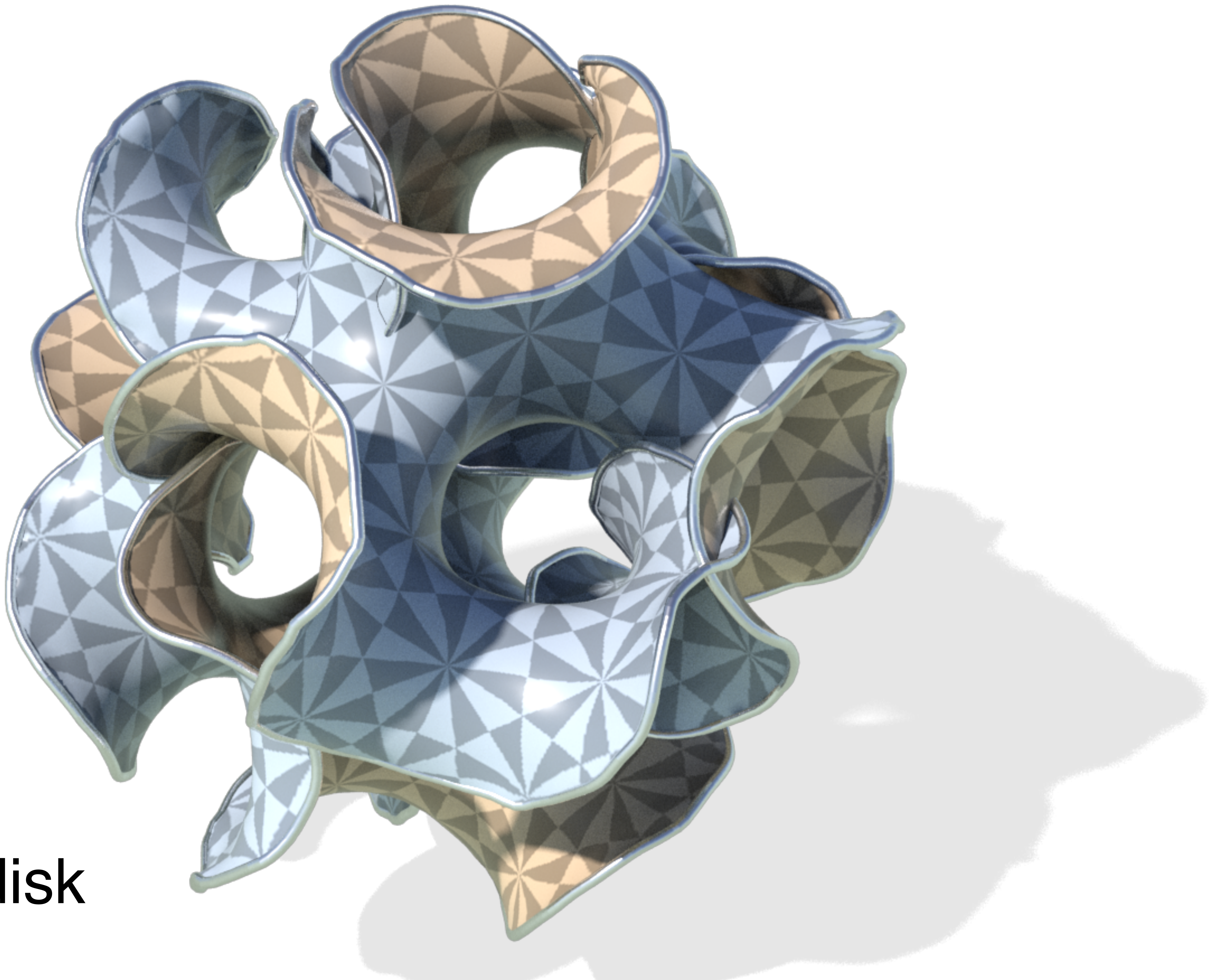

Hyperbolic disk

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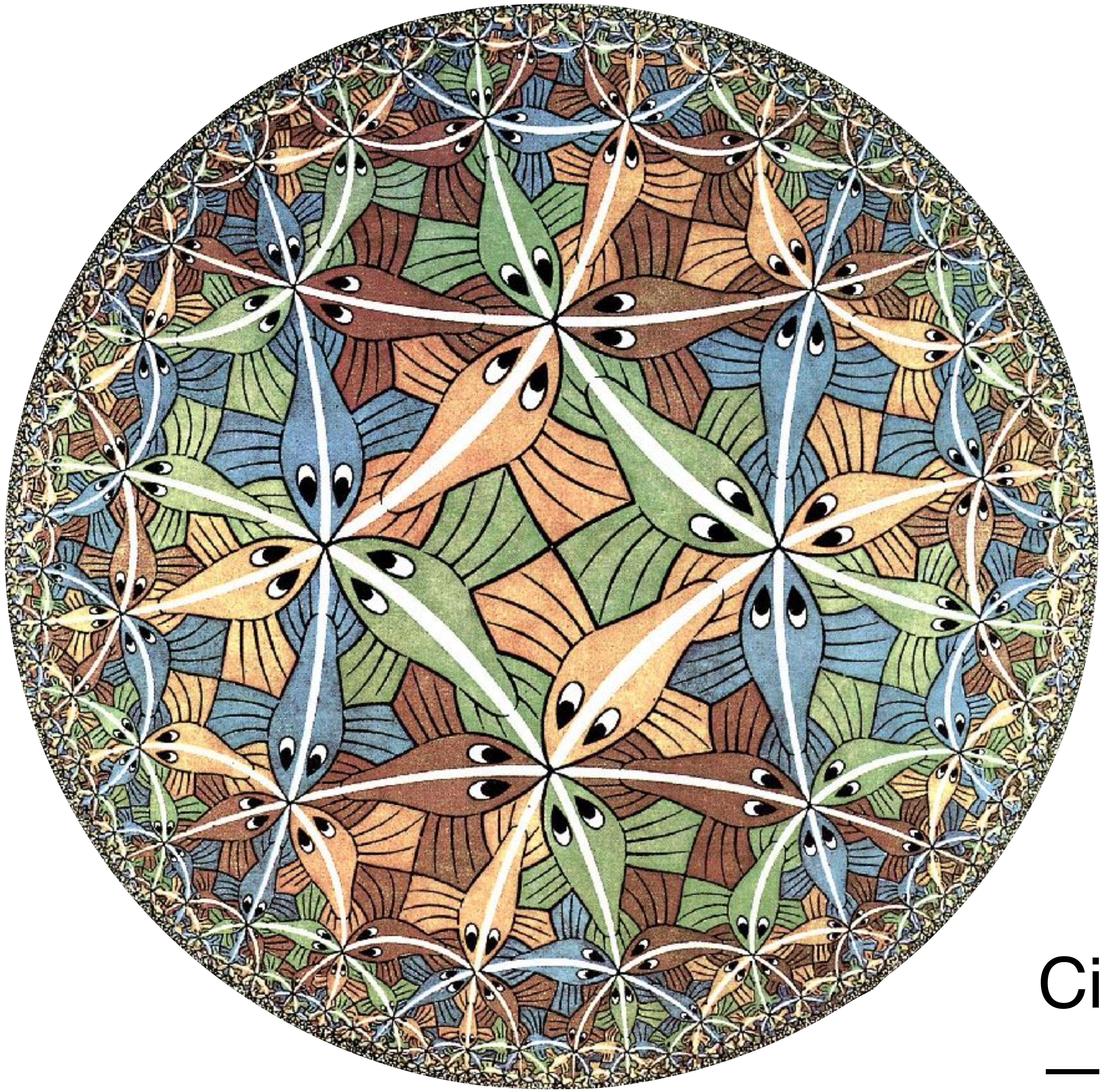
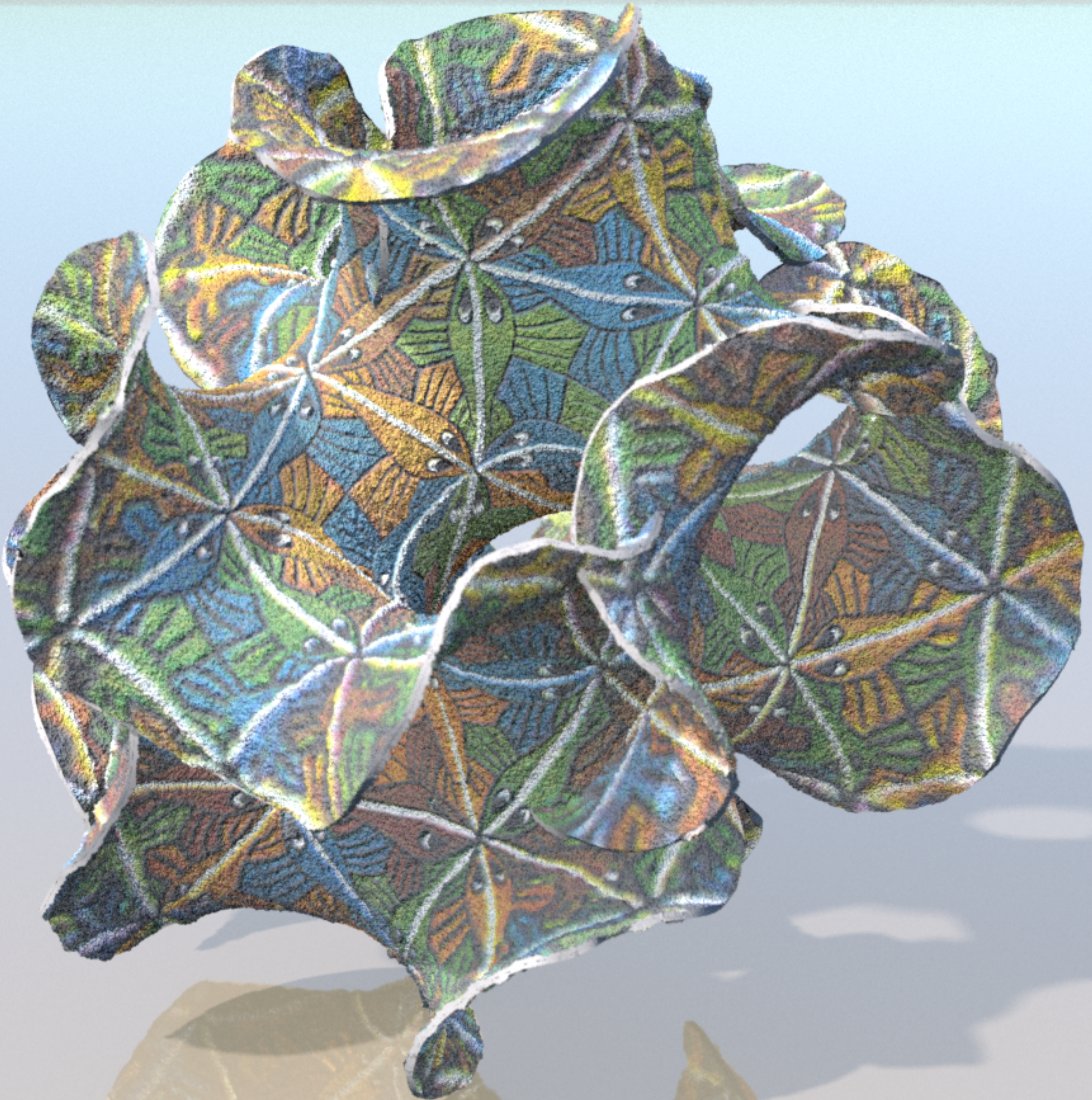
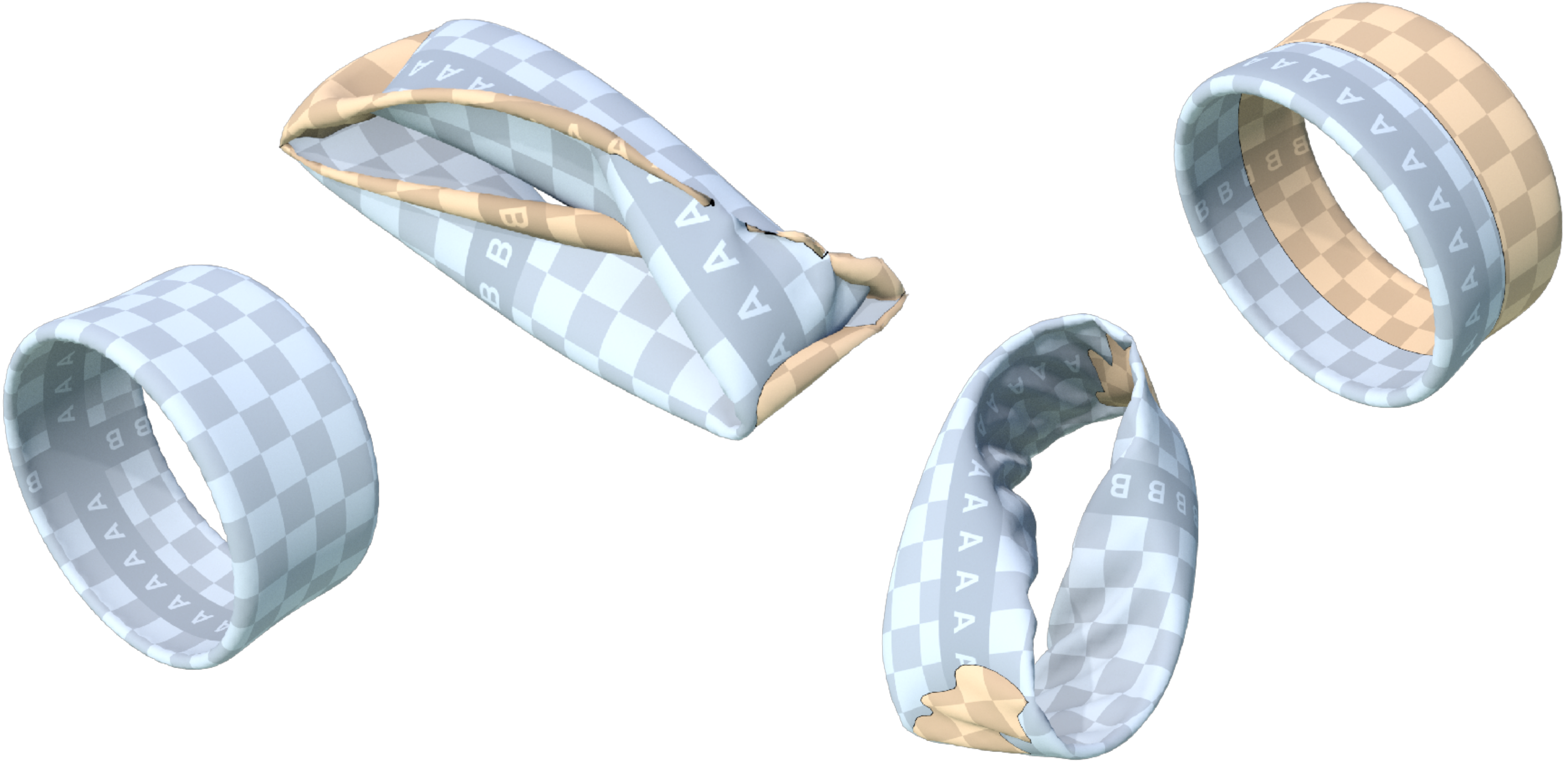


Circle Limit III
— M.C. Escher

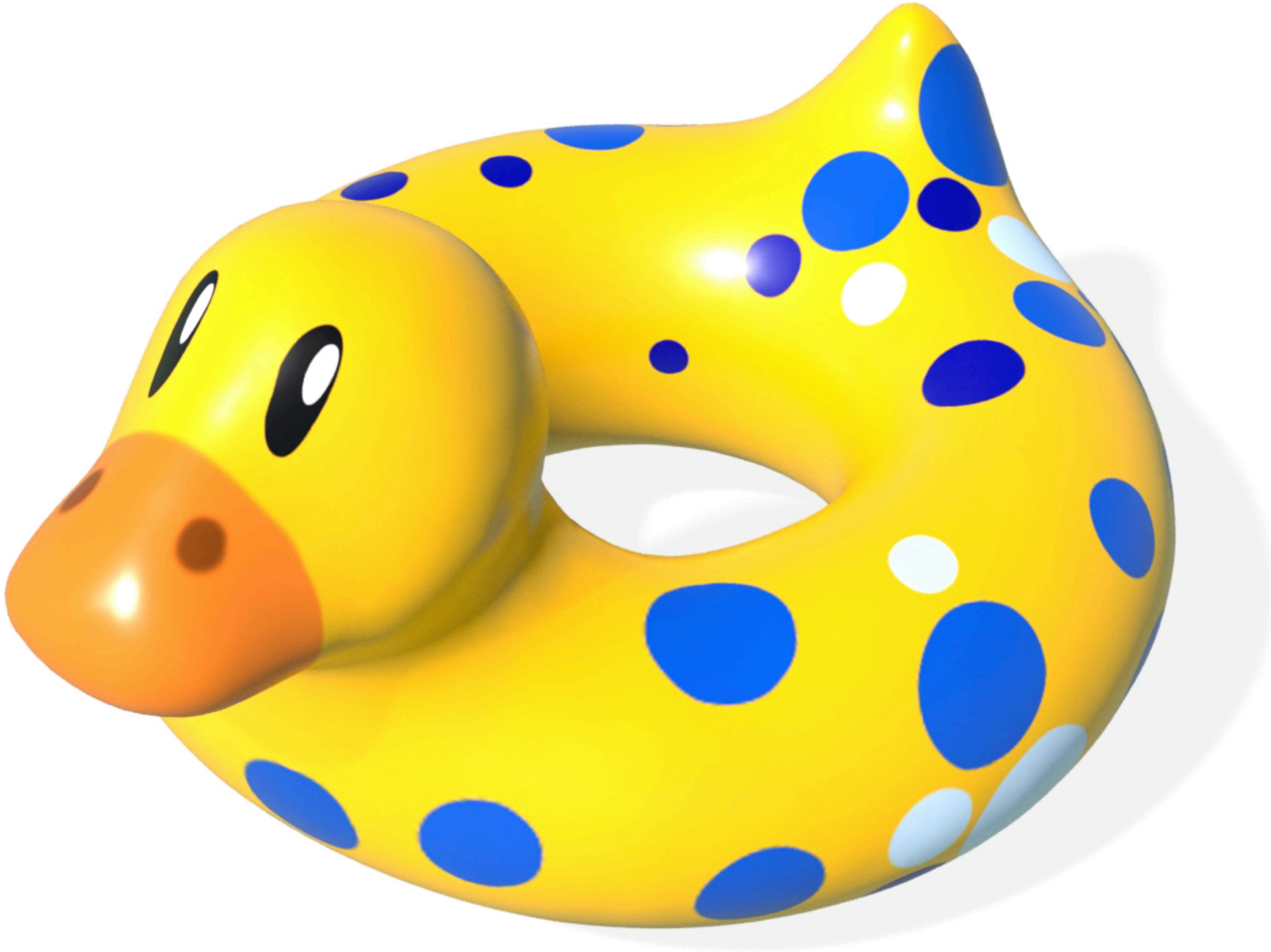
A Disk in Hyperbolic Plane



Flat Tori

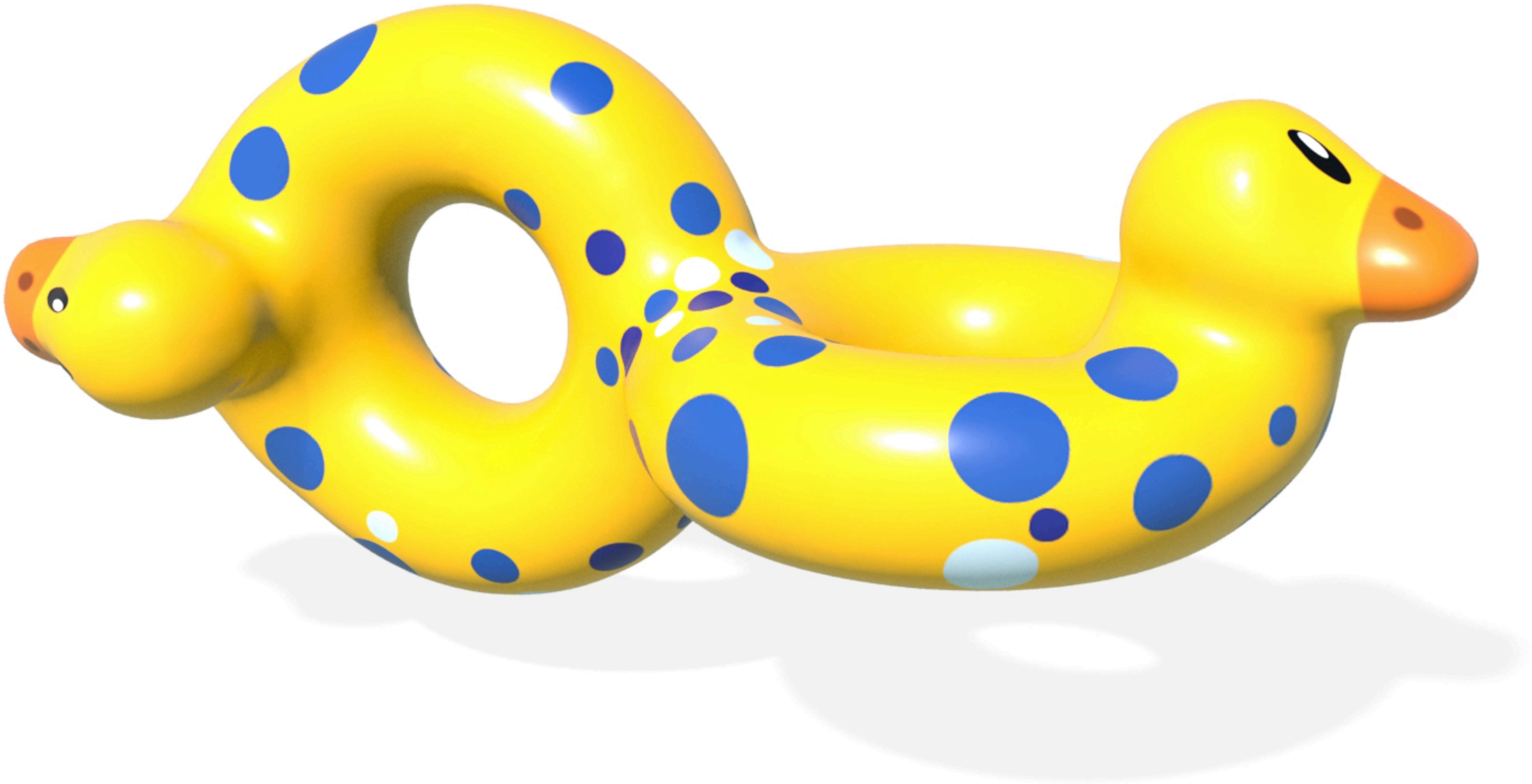


Visualizing Ricci Flow

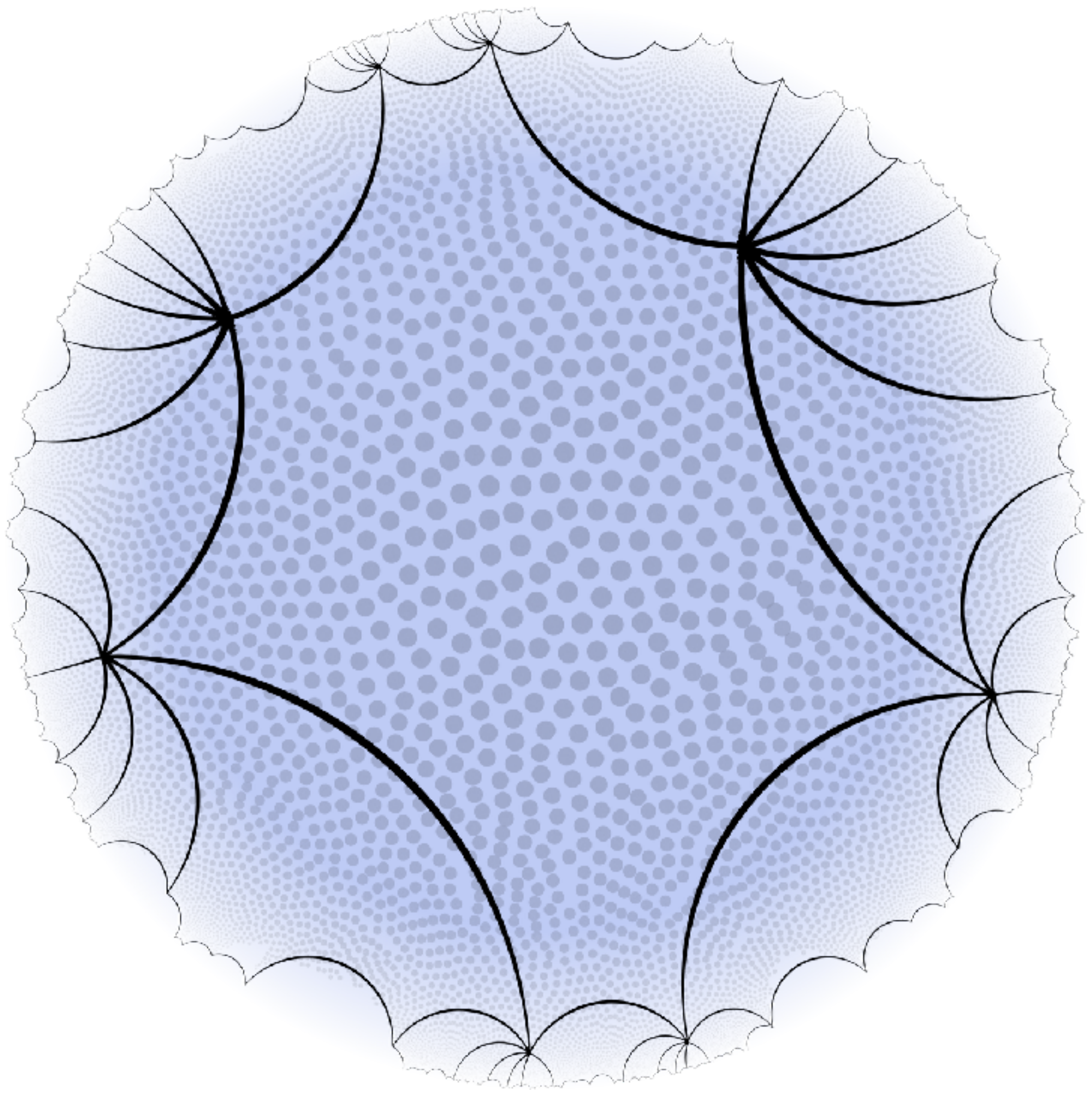


Metric modified by Ricci flow

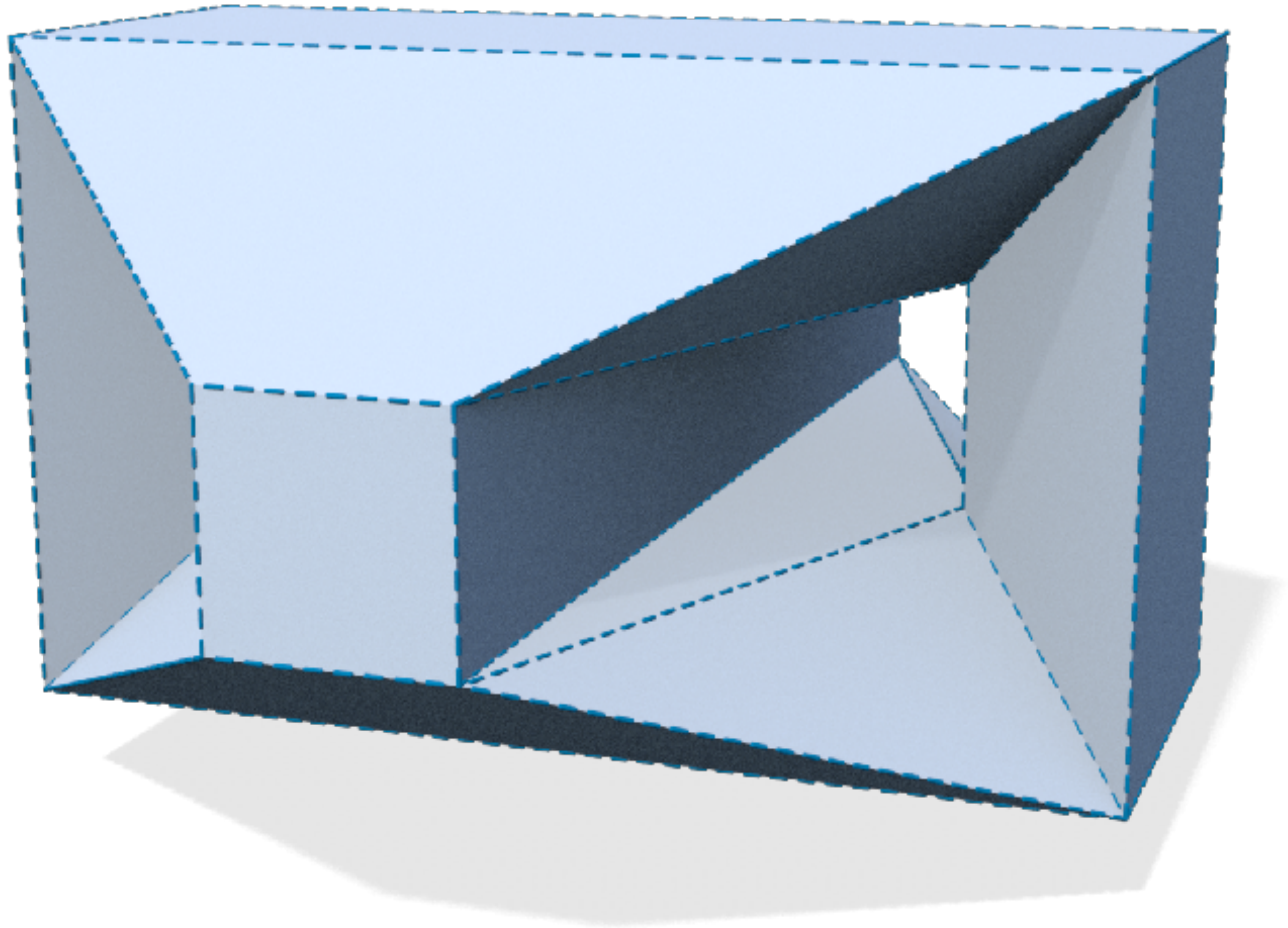
Constant negative curvature surface

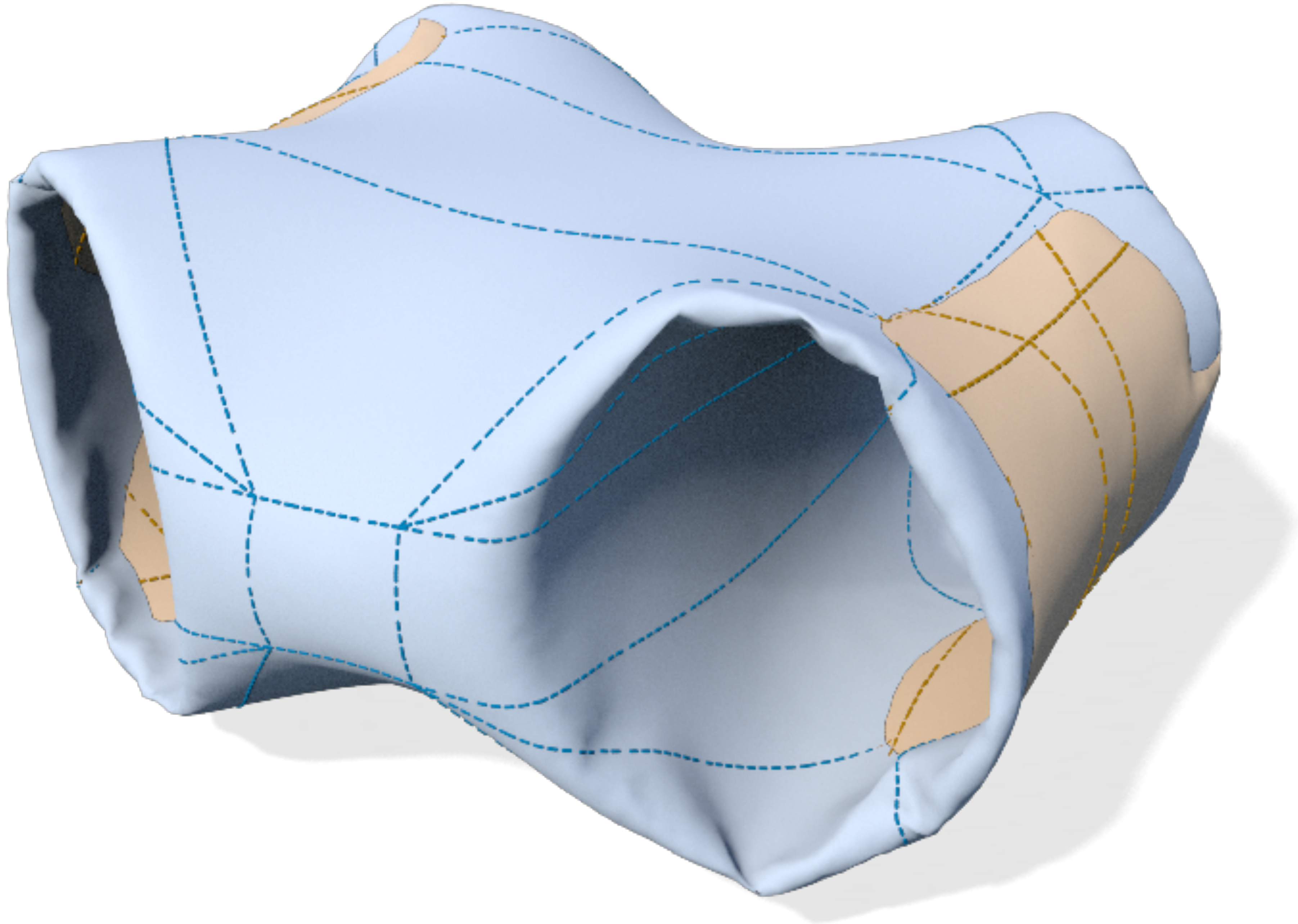


Random initial spinors



Constant negative gaussian curvature





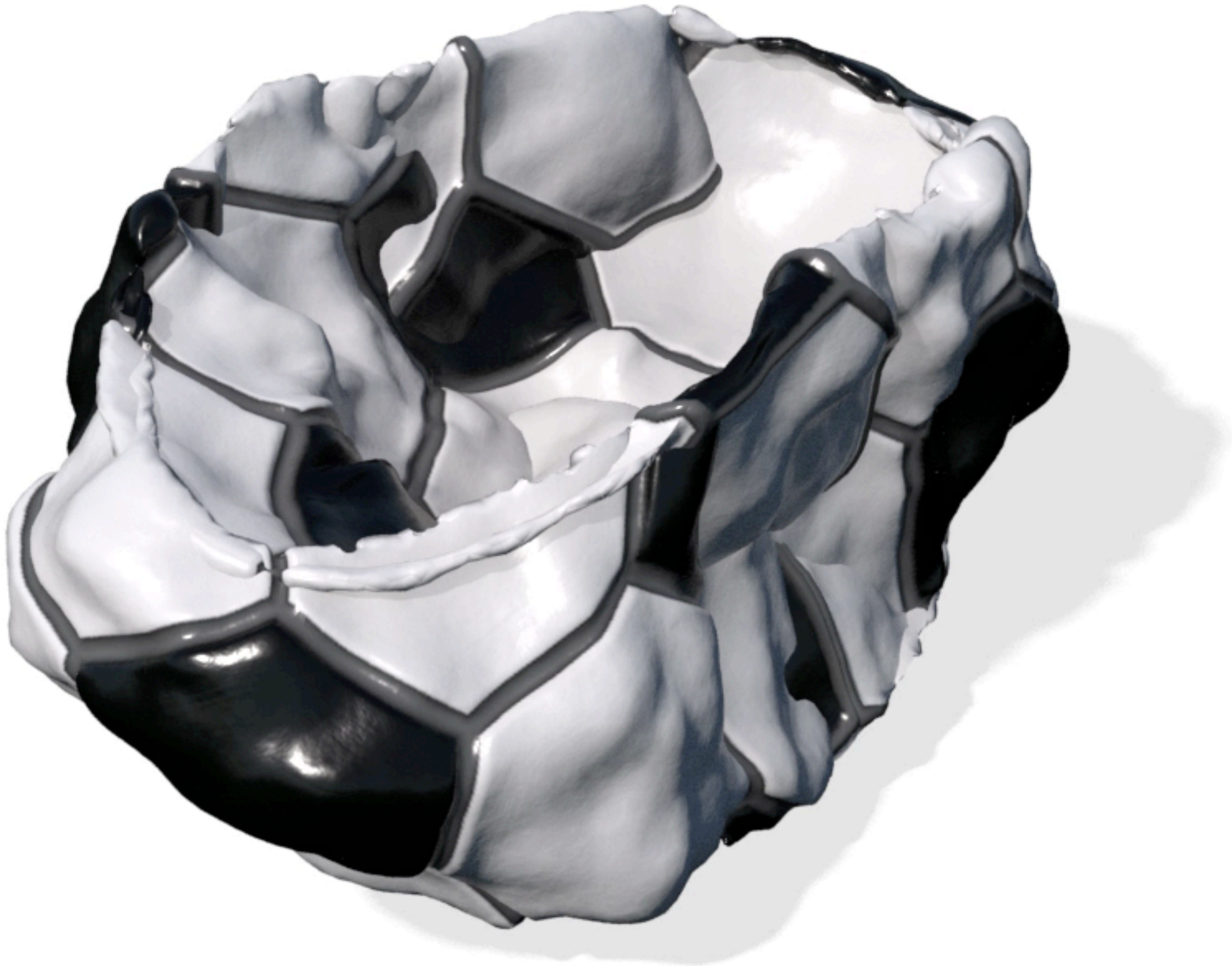
Piecewise-smooth isometric immersions

Conjecture

Each regular homotopy class of immersions of a 2D Riemannian manifold into $\mathbb{R}^3$ contains a piecewise smooth isometric representative.

Soccer ball

Soccer ball



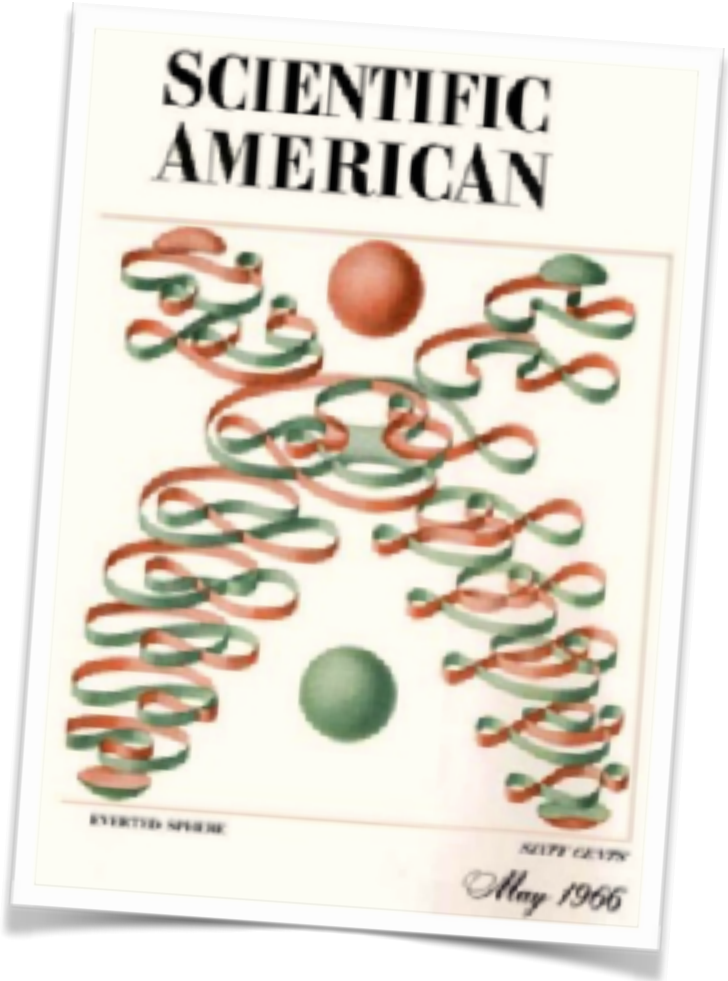
Soccer ball

Aluminium can

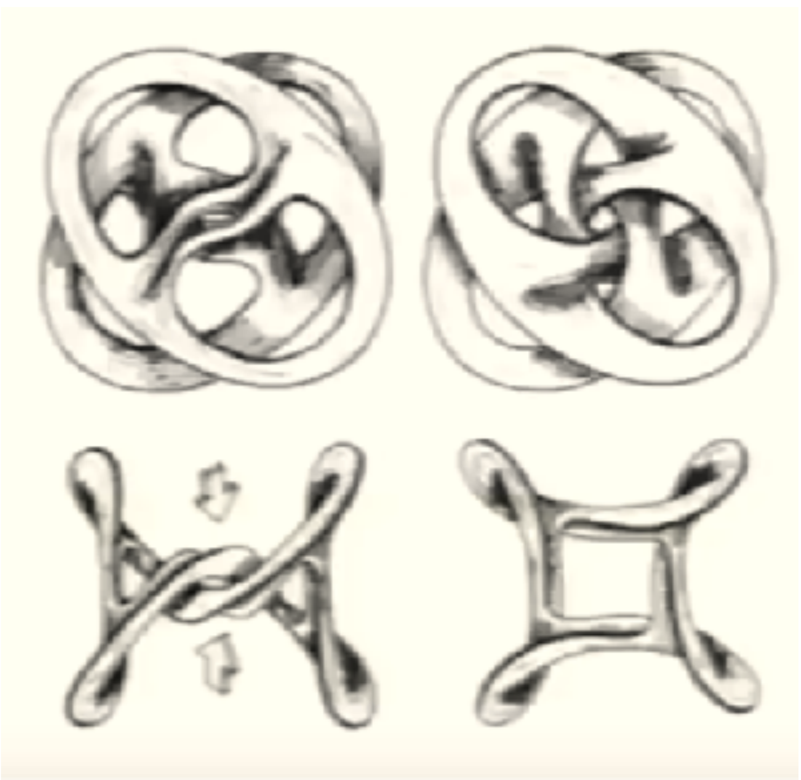
Aluminium can

Aluminium can

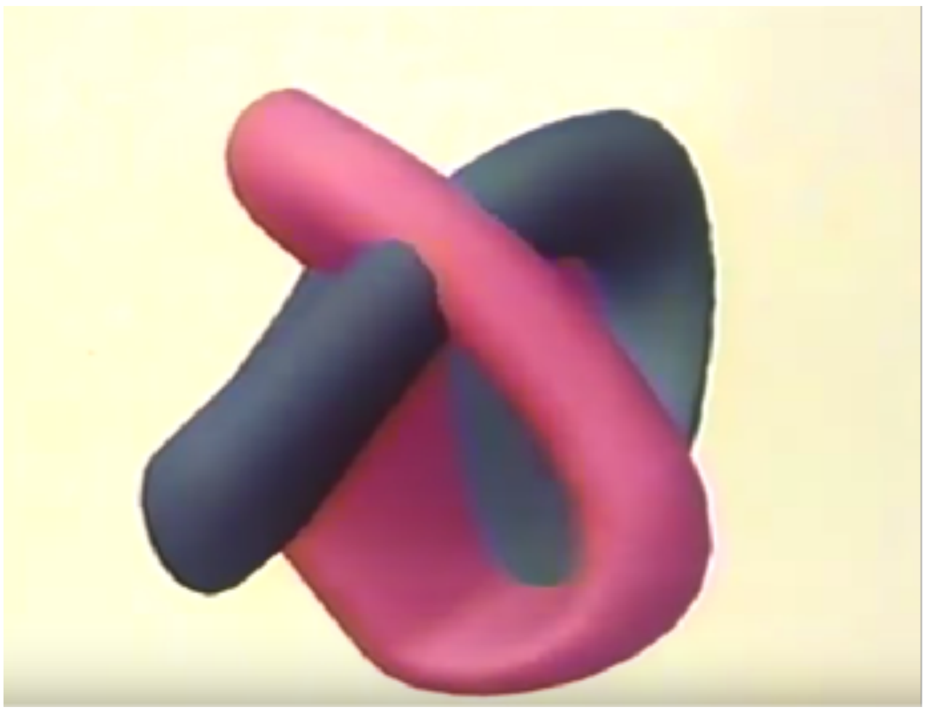
Turning the sphere inside out



A. Phillips
1966

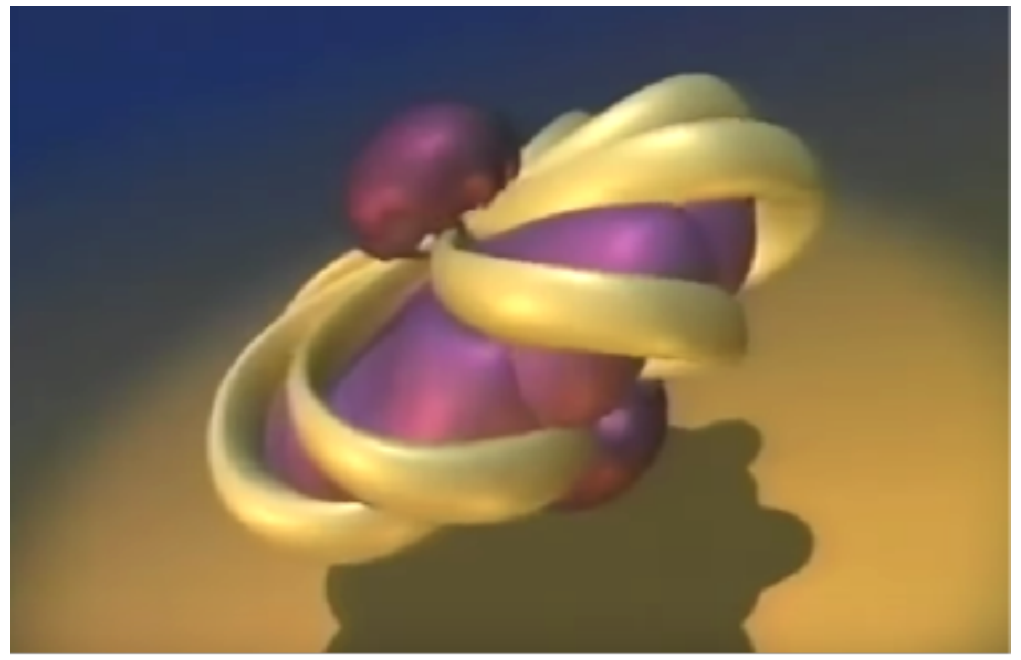


B. Morin, G. Francis
1967 / 1987



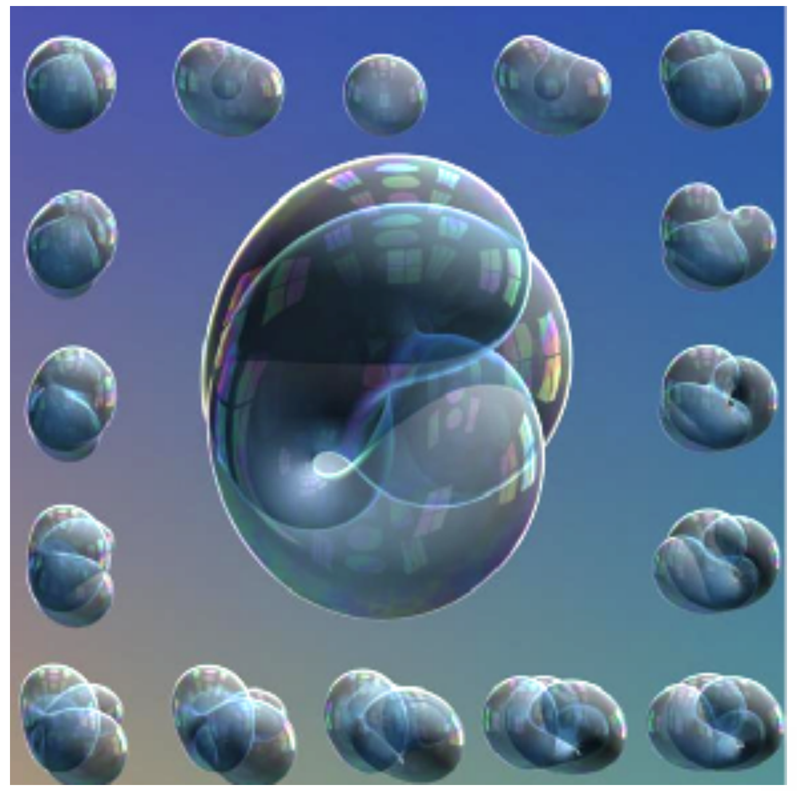
Film: *Turning a Sphere Inside Out*

N. Max 1977



Film: *Outside In*

W. Thurston,
S. Levy,
D. Maxwell,
T. Munzner 1994



Optiverse

J. Sullivan,
G. Francis,
S. Levy
1996

A. Chéritat
2014

Turn the bunny inside out isometrically

Thank You

- YouTube Shape from Metric
- Houdini implementations

Albert Chern
TU Berlin / UCSD

Felix Knöppel
TU Berlin

Franz Pedit
UMass Amherst

Ulrich Pinkall
TU Berlin

Peter Schröder
Caltech

"Shape from Metric"
ACM Trans. Graph. SIGGRAPH 2018

"Finding Conformal and Isometric Immersions of Surfaces"
arXiv: 1901.09432